Dynamic convex operators
Applications to Financial Markets

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INTRODUCTION

DYNAMIC RISK MEASURES
- Axiomatic
- Time Consistency and Cocycle Condition

RISK INDIFFERENCE PRICING
INTRODUCTION

In financial mathematics two main questions:

1. **EVALUATION OF RISKS**

Diversification $\rightarrow$ convexity
Not the case for $V@R$ often used.
Sublinear static risk measures first introduced by Artzner, Delbaen Eber Heath (2000)
Convex static risk measures Föllmer and Schied, Frittelli and Rosazza Gianin(2002)

2. **DYNAMIC PRICING OF FINANCIAL PRODUCTS**

In frictionless complete markets prices of financial assets are linear:
A no arbitrage dynamic price process: $E_Q(X|F_t)$
$Q$ equivalent to the reference probability measure: $Q$ Risk neutral probability measure
**INTRODUCTION**

**INCOMPLETE MARKETS**
No uniqueness of the risk neutral probability measure
Bid price $\neq$ Ask Price
Ask Price: surreplication price sublinear $\text{esssup}_{Q \in Q} E_Q(X|\mathcal{F}_t)$

Bid Price superlinear

**MARKETS WITH FRICTIONS, LIQUIDITY RISK**
Dynamic Ask Price is convex.
Dynamic Bid Price is concave.
OUTLINE

1 INTRODUCTION

2 DYNAMIC RISK MEASURES
   - Axiomatic
   - Time Consistency and Cocycle Condition

3 RISK INDIFFERENCE PRICING
**Dynamic Risk Measures**

**Framework**
Filtered probability space \((\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\).

\((\mathcal{F}_t)\) right continuous filtration

Modelize a dynamic risk measure \((\rho_{s,t}), 0 \leq s \leq t \leq T\).

\(\rho_{s,t} : L^p(\mathcal{F}_t) \rightarrow L^p(\mathcal{F}_s), 1 \leq p \leq \infty\)
DEFINITION

A dynamic risk measure $\rho_{s,t}$, $0 \leq s \leq t \leq T$ is a family of maps $\rho_{s,t} : L_p(\mathcal{F}_t) \rightarrow L_p(\mathcal{F}_s)$, satisfying:

1. **Monotonicity:**
   \[ \forall X, Y \in L_p(\mathcal{F}_t), \text{ if } X \leq Y \text{ then } \rho_{s,t}(X) \geq \rho_{s,t}(Y) \]

2. **Translation Invariance:**
   \[ \forall X \in L_p(\mathcal{F}_t), \forall Z \in L_p(\mathcal{F}_s) \rho_{s,t}(X + Z) = \rho_{s,t}(X) - Z \]

3. **Convexity:**
   \[ \forall (X, Y) \in L_p(\mathcal{F}_t), \forall \lambda \in [0, 1] \]
   \[ \rho_{s,t}(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_{s,t}(X) + (1 - \lambda)\rho_{s,t}(Y) \]
**Time Consistency**

**Definition**

A dynamic risk measure \( (\rho_{s,t})_{s,t}, \ 0 \leq s \leq t \leq T \) is time-consistent if

\[
\forall \ 0 \leq r \leq s \leq t \ \forall \ X \in L_p(\mathcal{F}_t) \quad \rho_{r,t}(X) = \rho_{r,s}(-\rho_{s,t}(X))
\]

**Continuity**

**Definition**

A dynamic risk measure \( \rho_{s,t}, \ 0 \leq s \leq t \leq T \)

1. is continuous from above if for any decreasing sequence \( X_n \) such that \( X = \lim X_n \ P \ a.s. \), the increasing sequence \( \rho_{s,t}(X_n) \) has the limit \( \rho_{s,t}(X) \) \( P \ a.s. \).

2. has the Fatou property if for every bounded sequence \( X_n \) converging to \( X \ P \ a.s. \), \( \rho_{st}(X) \leq \lim \inf \rho_{st}(X_n) \ P \ a.s. \).
**Dual Representation**

**Dual Representation of a Dynamic Risk Measure**

Let $\rho_{st}$ be a dynamic risk measure on $L_p$. Assume furthermore that $\rho_{st}$ is continuous from above in case $p = \infty$. Then

$$\forall X \in L_p(F_t) \quad \rho_{s,t}(X) = \text{esssup}_{R \in \mathcal{M}_{s,t}^1} (E_R(X|F_s) - \alpha_{s,t}^m(R)) \quad \text{P a.s.} \quad (1)$$

where $\mathcal{M}_{s,t}^1 = \{ R \text{ on } (\Omega, F_t), R \ll P, R|F_s = P \text{ and } E_R(\alpha_{s,t}^m(R)) < \infty \}$

$$\alpha_{s,t}^m(R) = \text{esssup}_{X \in L_p(F_t)} (E_R(X|F_s) - \rho_{s,t}(X)) \quad (3)$$

monetary case: Föllmer and Schied; also Frittelli and Rosazza Gianin
conditional case: Detlefsen and Scandolo and J. B.N.
Let \((\rho_s,t)_{0 \leq s \leq t \leq T}\) be a dynamic risk measure (continuous from above in case \(p = \infty\)). It is time-consistent if and only if for every probability measure \(Q \ll P\), the minimal penalty function satisfies the following cocycle condition:

\[
\forall r \leq s \leq t, \alpha^m_{r,t}(Q) = \alpha^m_{r,s}(Q) + E_Q(\alpha^m_{s,t}(Q)|\mathcal{F}_r) \quad Q \text{ a.s}
\]  

Characterization of time consistency in terms of cocycle condition, J. B.N. (2006)

Another characterization of time consistency was given in Cheridito, Delbaen, Kupper (2006) in terms of acceptance sets and also in terms of a concatenation condition.
**Theorem**

Let $Q$ be a stable set of probability measures all equivalent. Let $(\alpha_{s,t})$ be a penalty function defined on $Q$ satisfying the cocycle condition:

$$\forall Q \in Q, \forall r \leq s \leq t, \alpha_{r,t}(Q) = \alpha_{r,s}(Q) + E_Q(\alpha_{s,t}(Q)|\mathcal{F}_r)$$

Then

$$\rho_{s,t}(X) = \text{esssup}_{Q \in Q}(E_Q(-X|\mathcal{F}_s) - \alpha_{s,t}(Q))$$

defines a time-consistant dynamic risk measure continuous from above.
TIME-CONSISTENT DYNAMIC PRICE

Time-consistent dynamic ask price $\Pi_{s,t}$

$$\Pi_{s,t}(X) = \rho_{s,t}(-X)$$

Time-consistent dynamic bid price $\tilde{\Pi}_{s,t}$

$$\tilde{\Pi}_{s,t}(X) = -\Pi_{s,t}(-X)$$
1 INTRODUCTION

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3 RISK INDIFFERENCE PRICING
Risk Indifference Pricing first considered in 2007.

- In a static setting: “Risk indifference pricing in jump diffusion markets” by B. Oksendal and A. Sulem
- In a dynamic setting on $L_\infty(\mathcal{F}_T)$ for a fixed time horizon $T$. “Dynamic indifference valuation via convex risk measures” by Klöppel and Schweizer, making use of a Risk measure $\rho_t : L_\infty(\mathcal{F}_T) \rightarrow L_\infty(\mathcal{F}_t)$
- In a dynamic setting with respect to a dynamic risk measure $(\rho_{s,t})$ in $L_p$ spaces: joint work with Giulia di Nunno (2019).
**ADMISSIBLE STRATEGIES**

Market: underlying assets $(\Pi_t)_{t \in [0,T]}$ which are locally bounded semimartingales
Stable set $\Theta$ of admissible strategies on $[0, T]$

**FEASIBLE CLAIMS**

$$C^p_{st} := \{ g \in L_p(\mathcal{F}_t) : \exists \theta \in \Theta \text{ such that } g \leq Y_{s,t}(\theta) \}$$

From a qualitative perspective:

$$\text{essinf}_{g \in C^p_{s,t}} \rho_{s,t}(x_{s,t}(X) + y_s + g - X) = \text{essinf}_{g \in C^p_{s,t}} \rho_{s,t}(y_s + g) \quad P - a.s., \quad (5)$$

$y_s$: wealth at time $s$.

**ASSUMPTION**: $\text{essinf}_{g \in C^p_{s,t}} \rho_{s,t}(g) > -\infty \quad P - a.s.$
risk indifference pricing

**Definition**

Let \( p \in [1, \infty] \). Let \( \rho_{s,t} \) be a dynamic risk measure (continuous from above). For any \( s, t \in [0, T] : s \leq t \), the **Risk Indifference Price Operator**

\[
x_{s,t}(X) := \text{essinf}_{g \in \mathcal{C}^p_{s,t}} \rho_{s,t}(g - X) - \text{essinf}_{g \in \mathcal{C}^p_{s,t}} \rho_{s,t}(g)
\]

is well defined \( P \text{ a.s. for all } X \in L_p(\mathcal{F}_t) \) as an \( \mathcal{F}_s \)-measurable random variable.
PROPOSITION

The operator $x_{s,t}(X)$, $X \in L_p(F_t)$, is

- monotone, convex,
- it has the projection property: $x_{s,t}(X) = X$, $\forall X \in L_p(F_s)$
**PROPOSITION**

\[ x_{s,t}(X) \in L_\infty(\mathcal{F}_s) \text{ for all } X \in L_\infty(\mathcal{F}_t). \]

The restriction of \( x_{s,t} \) to \( L_\infty(\mathcal{F}_t) \) is a dynamic convex price operator (dynamic ask price) with values in \( L_\infty(\mathcal{F}_s) \).

It is continuous from above. It has the Fatou property on \( L_\infty \): For every bounded sequence \( X_n \) which converges \( P \) a.s. to \( X \),

\[ x_{s,t}(X) \leq \lim \inf_{n \to \infty} x_{s,t}(X_n). \]
However in general, for $p < \infty$, $x_{s,t}(X)$ does not belong to $L_p(\mathcal{F}_s)$ nor to $L_1(\mathcal{F}_s)$ for $X \in L_p(\mathcal{F}_t)$. $E(x_{st}(X))$ is not well defined on $L_p(\mathcal{F}_t)$. It is not a proper convex function. Furthermore $\text{Dom}(x_{s,t}) := \{X, x_{s,t}(X) \in L_p(\mathcal{F}_s)\}$ is not a Frechet lattice. We will construct an extension of $\rho_{s,t}$ and $x_{s,t}$ to the closure $L_t(c)$ of $L_p(\mathcal{F}_t)$ with respect to a $c$-norm. And we will prove that it takes values in $L_s(c)$. For this we assume hereafter that $\rho_{0,T}$ is dominated.
**Dominated Risk Measure**

In case $p < \infty$, we assume hereafter that the risk measure $\rho_{0,T}$ is *dominated*: there exists a sublinear (or coherent) risk measure $\tilde{\rho} : L_p(\mathcal{F}_T) \rightarrow \mathbb{R}$ such that

$$\forall X \in L_p(\mathcal{F}_T), \quad \rho_{0,T}(X) - \rho_{0,T}(0) \leq \tilde{\rho}(X).$$

**2 Motivations:** $\rho_{0,T}$ dominated

1. iff $\rho_{0,T}$ satisfies the sandwich condition:

   $$-\tilde{\rho}(-X) \leq \rho_{0,T}(X) - \rho_{0,T}(0) \leq \tilde{\rho}(X)$$

2. iff it admits a representation with a compact set of probability measures for the weak* topology (From “Risk measuring under model uncertainty” B.N. Kervarec 2012)

   $$\rho_{0,T}(X) = \max_{Q \in \mathcal{Q}} (E_Q(X) - \alpha(Q)), \quad \mathcal{Q} \subset \{Q \ll P, \|\frac{dQ}{dP}\|_q \leq K\}$$

   $q$ is the conjugate exponent of $p$ ($\frac{1}{p} + \frac{1}{q} = 1$).
The risk measure $\rho_{0,T}$ is sensitive (to $P$), i.e.

$$\rho_{0,T}(1_B) < \rho_{0,T}(0), \quad \forall B \in \mathcal{F}_T \text{ such that } P(B) > 0.$$ 

**Proposition**

Fix $p < \infty$. Assume that $\rho_{0,T}$ is dominated, and sensitive, then:

$$\rho_{0,T}(X) = \sup_{Q \in \mathcal{Q}} \left( E_Q(X) - \alpha_{0,T}(Q) \right), \quad X \in L_p(\mathcal{F}_T),$$

where $\mathcal{Q} := \left\{ Q \sim P : \alpha_{0,T}(Q) < \infty \right\}$.

Moreover, $\mathcal{Q} \subseteq \left\{ Q \sim P : \left\| \frac{dQ}{dP} \right\|_q \leq K \right\}$.
A **CAPACITY** $c$ on continuous bounded functions is a semi-norm $c$ satisfying:

$$c(f) \leq c(g), \quad \text{if } |f| \leq |g|$$

For every sequence $f_n$ decreasing to 0, $\inf_n c(f_n) = 0$

$$c(f) = \sup_{Q \in \mathcal{Q}} E_Q(|f|)$$

$\mathcal{Q}$: set of probability measures in the representation of $\rho_{0,T}$.

**DEFINITION**

$L_t(c)$ is the Banach space obtained from completion of the set of continuous bounded functions with respect to the seminorm $c$.

for all $Q \in \mathcal{Q}$:

$$L_p(\mathcal{F}_t, P) \subseteq L_t(c) \subseteq L_1(\mathcal{F}_t, Q).$$
REPRESENTATION OF $\rho_{s,t}$

$$\tilde{\mathcal{P}}_{s,t} := \left\{ R \sim P : R_{|\mathcal{F}_s} = P \text{ and } \sup_{Q \in \mathcal{Q}} E_Q(|\alpha_{s,t}(R)|) < \infty \right\}$$

PROPOSITION

Set $p \in [1, \infty)$. Let $(\rho_{s,t})_{s,t}$ be a strong time-consistent fully dynamic risk measure such that $\rho_{0,T}$ is dominated and sensitive. Then for all $t \in [0, T] : s \leq t$, the following representation holds:

$$\rho_{s,t}(X) = \operatorname{esssup}_{R \in \tilde{\mathcal{P}}_{s,t}} \left( E_R(-X|\mathcal{F}_s) - \alpha_{s,t}(R) \right), \quad X \in L_p(\mathcal{F}_t). \quad (8)$$
**PROPOSITION**

Let $s_0 = 0 < s_1 < \ldots < s_n = T$.

For all $Q_i$ in $\tilde{P}_{s_i,s_{i+1}}$, let $Q$ be the unique probability measure on $\mathcal{F}_T$ such that for all $X$ in $L_\infty(\mathcal{F}_T)$,

$$E_Q(X) = E_{Q_0}(E_{Q_1}(\ldots E_{Q_{n-1}}(X|\mathcal{F}_{s_{n-1}})\ldots|\mathcal{F}_{s_1})).$$

Then $Q$ belongs to $\mathcal{Q}$. 
.Extension of $\rho_{s,t}$ to $L_t(c)$

**Theorem**

Set $p \in [1, \infty)$. Let $(\rho_{s,t})_{s,t}$ be a strong time-consistent fully dynamic risk measure on $(L_p(F_t))_t$ such that $\rho_{0,T}$ is dominated and weak sensitive.

1. For all $0 \leq s \leq t \leq T$, $\rho_{s,t}$ admits a unique extension $\tilde{\rho}_{s,t}$ to $L_t(c)$ with values in $L_s(c)$ such that for all $X, Y \in L_t(c)$,

   $$c(|\tilde{\rho}_{s,t}(X) - \tilde{\rho}_{s,t}(Y)|) \leq c(|X - Y|).$$

2. The extension $\tilde{\rho}_{s,t}$ admits the following representation

   $$\tilde{\rho}_{s,t}(X) = \text{esssup}_{\mathcal{R} \in \tilde{\mathcal{P}}_{s,t}} (\mathbb{E}_{\mathcal{R}}(-X|\mathcal{F}_s) - \alpha_{s,t}(\mathcal{R})), \quad X \in L_t(c).$$

3. $(\tilde{\rho}_{s,t})_{0 \leq s \leq t \leq T}$ is a strong time-consistent dynamic risk measure on $(L_t(c))$. Furthermore $\tilde{\rho}_{0,T}$ is dominated by $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q(-X)$.
**DEFINITION**

Let $0 \leq s \leq t \leq T$. For all $X \in L_t(c)$, define

$$x_{st}(X) := \underset{g \in C^p_{s,t}}{\text{essinf}} \tilde{\rho}_{s,t}(g - X) - \underset{g \in C^p_{s,t}}{\text{essinf}} \tilde{\rho}_{s,t}(g),$$

where $\tilde{\rho}_{s,t}$ is the extension of $\rho_{s,t}$ to $L_t(c)$.
**PROPOSITION**

1. $x_{s,t}$ is a well defined operator on $L_t(c)$ with values in $L_s(c)$ extending the risk indifference price operator previously defined on $L_p(F_t)$. Moreover, for all $X, Y \in L_t(c)$,

   \[
   c(|x_{s,t}(X) - x_{s,t}(Y)|) \leq c(|X - Y|) \quad (11)
   \]

2. $x_{s,t}$ is convex monotone translation invariant and normalized.
PROPOSITION

Let $p < \infty$. Let $(\rho_{s,t})_{0 \leq s \leq t \leq T}$ be a strong time-consistent fully dynamic risk measure on $L_p(\mathcal{F}_t)$. Assume that $\rho_{0,T}$ is dominated and sensitive. Let $0 \leq r \leq s \leq t \leq T$. Let $(x_{s,t})_{0 \leq s \leq t \leq T}$ be the risk indifference price extended to $L_t(c)$. The operator $x_{s,t}$ has the FATOU PROPERTY on $L_t(c)$: For any sequence $X_n$ in $L_t(c)$, dominated in $L_t(c)$ and converging $P$ a.s. to $X$, with $X \in L_t(c)$,

$$x_{s,t}(X) \leq \liminf_{n \to \infty} x_{s,t}(X_n)$$  \hspace{1cm} (12)

$X_n$ dominated in $L_t(c)$ means that there is $Y \in L_t(c)$ such that $|X_n| \leq Y$ $P$ a.s. for all $n$. 

Let $p < \infty$. Let $(\rho_{s,t})_{0 \leq s \leq t \leq T}$ be a time-consistent dynamic risk measure on $L^p(F_t)$. Assume that $\rho_{0,T}$ is dominated and sensitive. Let $0 \leq r \leq s \leq t \leq T$. The risk-indifferent dynamic price $(x_{s,t})_{s \leq t}$ satisfies the following time-consistency property on the whole $L_t(c)$. Let $X, Y \in L_t(c)$ such that $x_{s,t}(X) = x_{s,t}(Y)$, then $x_{r,t}(X) = x_{r,t}(Y)$.
CONCLUSION

We have studied the properties of dynamic convex operators in $L_p$ spaces $(1 \leq p \leq \infty)$
Applications to dynamic risk measures and dynamic pricing.
We have defined the dynamic risk indifference price $(x_{s,t})_{0 \leq s \leq t \leq T}$ associated to a time-consistent dynamic risk measure $(\rho_{s,t})_{0 \leq s \leq t \leq T}$ on $L_p$.

- In case $p = \infty$, $x_{s,t}$ is a pricing system on $L_\infty$.
- In case $p < \infty$, we assume that $\rho_{0,T}$ is dominated and sensitive. We have introduced a capacity $c(X) = \sup_{Q \in Q} E_Q(|X|)$ related to the dominated risk measure $\rho_{0,T}$. We have proved that the operators $\rho_{s,t}$ and $x_{s,t}$ admit a unique continuous extension to $L_t(c)$, and that $(L_t(c))_{0 \leq t \leq T}$ is the right setting for the study of dynamic risk indifference pricing. Time consistency and Fatou property are proved.
MERCI POUR VOTRE ATTENTION