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A NOTE ON COMPARING MEDIAN EVALUATIONS IN SINGLE-PEAKED DOMAINS

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A note on comparing median evaluations in single-peaked domains

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Abstract

In one-dimensional, single-peaked domains, the paper compares the MaxMedian voting scheme of Basset and Persky (*Public Choice* 99: 299-310) with majority rule and the utilitarian criterion. The MaxMedian outcome is rejected by a majority of voters in favor of outcomes which are also utilitarian improvements.

1 Introduction

When different individuals evaluate an object on some common scale, taking as a summary of the various evaluations their median (if well defined) is a natural idea, often used in Statistics. Using this idea for collective choice among several alternative objects is thus occasionally proposed: the suggestion is that an alternative with the largest median evaluation is to be chosen. I will call this method the *MaxMedian* voting scheme.

In the modern literature this was (up to my knowledge) proposed by Basset and Persky, 1999 [2]. Properties of the best median can be expressed in the language of Social Choice Theory. The informational basis of the median is the ordinal and inter-individuallly comparable framework. In terms of utilities, the utility levels attached by the different individuals to a given alternative are compared (inter-individual comparability), and the median is stable by any strictly increasing transformation of utility provided that the transformation is the same for all individuals (ordinality). This is the same informational basis as for the MaxMin, or Rawls principle. Instead of maximizing the satisfaction of the least favored individual in the whole society, as the MaxMin does, MaxMedian voting maximizes the satisfaction of the least favored individual within half of the population.

As to strategic aspects, supporters of this method claim that it is relatively immune to individual misrepresentation of evaluations. The reason for this claim is that the median (contrary to the mean) is "robust" in the statistical sense. For instance, in most cases, over-evaluating an object in order to push up its median has simply no consequence. This argument was put forward by Basset and Persky [2], who used the term "Robust Voting" to describe this method. But, in fact, voting rules generally share the property that one voter is rarely pivotal so that, in practice, median voting does not appear to be less, nor more, manipulable than other voting rules (Gerhlein and Lepelley 2003 [4]).

Properties of the method are presented on the web by the Center for Range Voting [6]. Using a limited set of grades, several candidates usually end up with the same grade. Balinski and Laraki, 2007 [1] have proposed several, more or less complicated, ways to choose among them, and have ellaborated on the question of robustness to manipulation. Felsenthal and Machover, 2008 [3] is a discussion of these issues.

In this paper I will leave aside the strategic questions and assume that voters vote sincerely. I also leave aside the problem of ties associated with the use of MaxMedian voting in practice, by assuming that an evaluation can be any real number. I will attempt to compare the outcomes of different choice principles, including MaxMedian, in a setting which is standard in economic and political theory and is relevant for the applications. I consider one-dimensional, single-peaked utility profiles with distributions of voters' ideal points which are skewed in one direction.

From the point of view of Social Choice Theory, one-dimensional, single peaked profiles are very specific profiles because they avoid Condorcet cycles, but they nevertheless constitute an interesting benchmark case. In Political Economy, this assumption is so common that it is usually not even mentioned. The idea that the distribution of voters with respect to the relevant parameter is not uniform or symmetric but skewed is an empirical observation: it seems to be a general rule that socioeconomic relevant parameters are "skewed to the left," the paradigmatic example being income distributions: most people earn less than the average.

In this setting, Condorcet-consistent voting rules are very simple, and all alike: all of them chose the median of the voters' ideal points, which is a *Condorcet* winner in virtue of the celebrated Median Voter Theorem. This is the outcome of majority voting. By comparison, the utilitarian choice (the efficient alternative in the sense of maximizing the sum of individual utilities, I will call it the *Bentham* winner) tends, in the same setting, to produce choice which are favored by richer people. Although this observation is not as clear-cut as the Median Voter Theorem is, it matches the economic intuition and can be stated formally if one makes the (standard) assumption of quadratic utility. In that case the utilitarian optimum is simply the mean of the distribution of ideal points. For left-skewed distribution, the mean is larger than the median.

Then where is the alternative with the best median evaluation located ? Is it close to the utilitarian optimum or not, and if not, in which direction does it diverges from the optimum: to the left, in the direction of the ideal points of the majority, or to the right, in the direction of the rich minority ? How does it compare to the outcome of Majority rule, the Condorcet winner ? Is it more or less efficient than this outcome ?

In order to answer this question, I present one very simple analytical example, plus computer simulations under various hypothesis which will be described in the sequel. I tried to stick to reasonable models, relatively close to "real worlds." The reached conclusion is always the same: the best median is located on the wrong side of the Condorcet winner. The Bentham-inneficiency of the best median choice is of the same kind but worse than the Bentham-inneficiency of majority voting.

The example maybe useful in order to understand what one does when comparing medians. The usual argument for rejecting an alternative A in favor of another alternative B when a majority of individuals prefer A to B is that members the relatively small population who gain in this move gain a lot while members of the losing majority incurs a relatively small loss. This is a typical Benthamite, utilitarian argument. Conversely, the Bentham-inneficiency of majority voting derives from the democratic power of the (many) poor[1]. This inefficiency can be justified by normative political arguments in favor of the principle of majority rule. It can also often be justified by invoking the argument of decreasing individual marginal utility. In the one-dimensional settings under scrutiny, the best median choice goes further away from efficiency, for reasons that are easily understood from the mathematical point of view (the basic example in section 3 will explain this point) but which have no normative or political appeal: apply Rawls principle to the most homogeneous half-population, with no regard for the other half.

2 Definitions and notation

The set of available alternatives is the set of positive real numbers $X = [0, +\infty[$. To each individual i is attached an ideal point $x_i \in X$. The utility of i for an alternative $y \in X$ is denoted $u_i(y)$. Under the quadratic utility assumption one has

$$u_i(y) = -(y - x_i)^2 .$$

(Variants of this assumption will be considered later.)

The society is then described by the distribution of ideal points. Let F denotes the cumulative distribution: for any $x \in X$, $F(x)$ is the proportion in the society of individuals i such that $x_i < x$. We suppose that F is continuous and has a mean. As it is well known, the Condorcet winner is then well-defined and unique, it is the median of the distribution, that is the alternative, denoted x_{Cond} such that

$$F(x_{\text{Cond}}) = 1/2.$$

The alternative which maximizes the total utility $W(y) = \int_X -(y-x)^2 dF(x)$ is the average of the ideal points. We call this point the Bentham optimum and

[1] At the limit one can wonder why, under majority rule, the poor do not expropriate the rich, see Roemer, 1998 [5].

denote it by x_{Bentham}:

$$x_{\text{Bentham}} = \int_x xF(x)dx$$

For any alternative y, the utility levels obtained by the various individuals induce a probability distribution that we denote by E_y : for any utility level $h \leq 0$, $E_y(h)$ is the proportion of individuals in the society whose utility for y is less than h. Given our assumption about utilities, these individuals are precisely those whose ideal points are at a distance larger than $\sqrt{-h}$ from y:

$$\begin{aligned} u_i(y) < h &\iff -(y - x_i)^2 < h \\ &\iff x_i \notin \left[y - \sqrt{-h}, y + \sqrt{-h}\right] \end{aligned}$$

thus

$$E_y(h) = 1 - F(y + \sqrt{-h}) + F(y - \sqrt{-h})$$

and the median evaluation of y, denoted $h_{\text{med}}(y)$ is such that $E_y(h_{\text{med}}(y)) = 1/2$, that is:

$$F(y + \sqrt{-h_{\text{med}}(y)}) - F(y - \sqrt{-h_{\text{med}}(y)}) = 1/2.$$

It may be more convenient to write this formula with $d(y) = \sqrt{-h_{\text{med}}(y)}$ as:

$$\int_{y-d(y)}^{y+d(y)} dF(x) = 1/2.$$

The best median evaluation choice, the outcome of "median evaluation" is denoted by x_{MV}, its is the point which maximizes h_{med} or, equivalently, which minimize d. In general it is uneasy to compute this point, but the example in the next section makes these computations very simple.

3 A basic example

Suppose that the distribution of voters' ideal point is triangular. The support of the distribution is $[0, 2]$ and the density function is $f(x) = 1 - x/2$. Then for $0 \leq x \leq 2$,

$$F(x) = \int_0^x (1 - t/2)\, dt = x - x^2/4.$$

Majority Voting: Condorcet winner. See point C in Figure 1. The median of the ideal points is such that $F(x_{\text{Cond}}) = 1/2$, that is

$$x_{\text{Cond}} = 2 - \sqrt{2} \simeq 0.586.$$

Utilitarian evaluation: Bentham optimum. See point B in Figure 1. The average of the ideal points is

$$x_{\text{Bentham}} = \int_0^2 t\,(1 - t/2)\, dt = 2/3 \simeq .667.$$

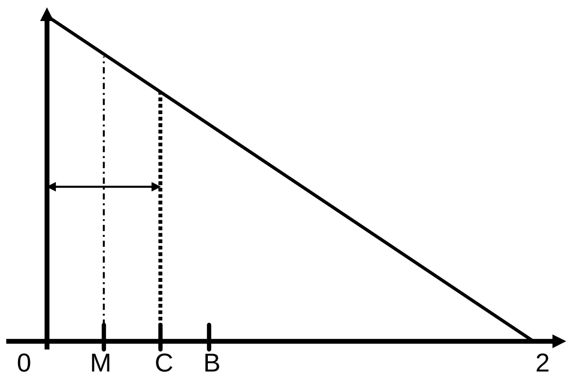

Figure 1: Three choices from a triangular distribution of ideal points

Median Evaluation: Max Median. See point M in Figure 1. In order to compute the median evaluation at a point y one has to find a radius $d(y)$ such that half of the ideal points are located in the segment $[y - d(y), y + d(y)]$, and half are located outside. The best median is obtained at the point y such that $d(y)$ is the smallest. In Figure 1, one can see that the best median is precisely the mid-point between 0 and the Condorcet winner x_{Cond}. Indeed, by definition of x_{Cond}, half of the population belongs to the segment $[0, x_{\text{Cond}}]$, and because the density is decreasing, any other segment of length smaller or equal will contain strictly less than half of the population. Therefore:

$$x_{\text{MV}} = 1 - \sqrt{2}/2 \simeq 0.293.$$

One can see on this basic example the result announced in the introduction:

$$x_{\text{MV}} < x_{\text{Cond}} < x_{\text{Bentham}}$$

and one explanation of the phenomenon. Half of the population lies on each side of the Condorcet winner, but the right window (between C and 2) is wider than the left one (between 0 and C). This should be an argument in favor of a collective choice larger than C, like the utilitarian choice B, because individuals at the left of C are relatively close to C while those at the right of C are relatively far from C. This is a typical utilitarian argument, that weights numbers of individuals and intensity of preferences. The reasoning that leads to the choice

M is reversed: the window being narrower on the left and wider on the right, choosing a point in the middle of the left window will satisfy the left half of the population and will reach for the — already quite satisfied — members of this group a relatively high level of satisfaction because this group is not too diverse. But if choosing a point on the right of M, such as C or B, the satisfied half of the population will be spread over a larger segment and it will be more difficult to reach the same level of satisfaction for this half of the population, because these people will be more diverse.

The choice of M is dictated by the level of satisfaction obtained by some half of the population. By definition of the MaxMedian, this level is the largest than can be obtained by any half of the population, thanks to inter-individual comparability. But, doing so, it neglects the (maybe very low) level of satisfaction obtained by the other half of the population, thanks to ordinality. There is no compromise here: find the half of the population which would be better off if they were in power ! This "majoritarian" logic is flawed, as one can see on the example, because the result is that moves away from M in the direction of the center simultaneously (i) satisfy the majority criterion because are preferred my most voters (more losers than winners) and (ii) satisfy the utilitarian criterion that the losers loose less than what the winners win.

4 Simulated examples with Log-Normal distribution

In the simulations, I compute the Bentham optimum, the outcome of Majority Rule (the Concorcet winner), the outcome of the Borda rule (the Borda winner) and the point with the best median evaluation. I work with a Log-Normal distribution, which is typically the kind of distribution met in Social Sciences (Figure 2). The theoretical mean of the Log-Normal distribution of parameters 0 and 1 is 1.649 and the standard deviation is 2.161. The distribution vanishes quickly after $x = 5$ and, when needed, I restrict attention to the segment $[0, 5]$. I pick at random 999 ideal points according to this distribution. Then, in a first example, the individual utility functions are *quadratic*: $u_i(y) = -(y - x_i)^2$, and there are 11 candidates evenly spread between 0 and 5 (at points 0, .5, 1, 1.5, $2, 2.5$, ..., 5). In that case, the Bentham optimum is located at $x_{\text{Bentham}} = 2$, the Borda winner and the Condorcet winner are both located at $x_{\text{Cond}} = x_{\text{Borda}} = 1$, and the best median evaluation is obtained at 0.5, thus: $x_{\text{MV}} < x_{\text{Cond}} = x_{\text{Borda}} < x_{\text{Bentham}}$

An alternative specification is that the utility is decreasing linearly with the distance: $u_i(y) = -|y - x_i|$. I label this case *linear* utility. Going from one specification to the other does not change the Condorcet or Borda winner nor does it change the median evaluations, because this is a strictly increasing transformation common to all individuals. But it changes the utilitarian optimum, which is now equal to the Condorcet winner. (Recall that the solution to the problem $\min_y \sum_i |y - x_i|$ is the median of the x_is.). Thus in that case, one has:

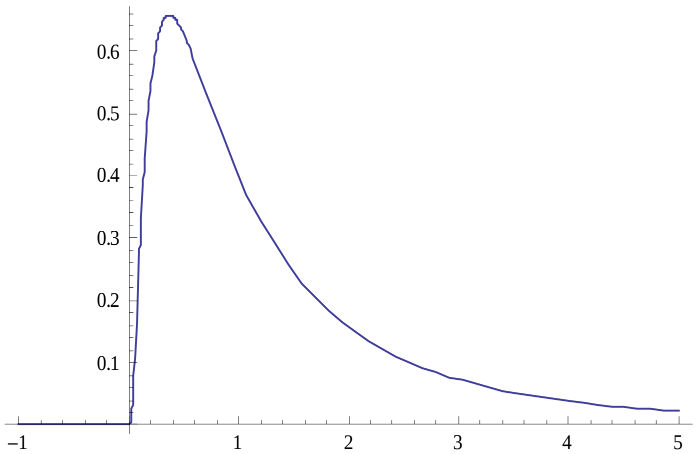

Figure 2: Log-Normal distribution

$x_{\mathrm{MV}} < x_{\mathrm{Cond}} = x_{\mathrm{Borda}} = x_{\mathrm{Bentham}}$.

Although economists usually work with concave utility functions (used as VNM utilities, these function are risk-adverse), it also makes sense in Politics to consider that the marginal utility is decreasing with the distance to the ideal point and to use utilities of the form $u_i(y) = -\sqrt{|y - x_i|}$. We label this case *root* utility. In the example, simulation shows: $x_{\mathrm{MV}} < x_{\mathrm{Cond}} = x_{\mathrm{Borda}} = x_{\mathrm{Bentham}}$.

Table 1 reports the results of simulations in various cases. Quadratic utility is used for the lines labelled $-d^2$, linear utility for the lines $-d$, and root utility for the lines $-d^{1/2}$. The number of candidates, n_k, is 11 or 49. Candidates are either evenly spaced from 0 to 5 ("uniform" case), or chosen according to the same probability distribution of the voters ("representative" case).

The simulations reported in Table 1 are not averaged. Randomness comes from the choice of the 999 individual ideal points and (in the "representative" setting) of the choice of the candidate positions. This last point is important for $n_k = 11$. In order to check the robustness of the results I replicated some of the above experiences. For instance here are the results obtained during 100 simulations for the representative case with $n_k = 11$ candidates and quadratic utilities.

Out of 100 simulations, the usual ranking is: $x_{\mathrm{MV}} < x_{\mathrm{Borda}} \leq x_{\mathrm{Cond}} < x_{\mathrm{Bentham}}$. More exactly:

- the strict inequality $x_{\mathrm{MV}} < x_{\mathrm{Borda}}$ is seen 82 times, $x_{\mathrm{MV}} = x_{\mathrm{Cond}}$ is seen 17 times, and $x_{\mathrm{MV}} > x_{\mathrm{Cond}}$ is seen only once;
- the strict inequality $x_{\mathrm{Borda}} < x_{\mathrm{Cond}}$ is seen 50 times, $x_{\mathrm{Borda}} = x_{\mathrm{Cond}}$ is seen 46 times, and $x_{\mathrm{Borda}} > x_{\mathrm{Cond}}$ is seen 4 times;
- the strict inequality $x_{\mathrm{Cond}} < x_{\mathrm{Bentham}}$ is seen 95 times, $x_{\mathrm{Cond}} = x_D$ is seen 5 times, and $x_{\mathrm{Cond}} > x_{\mathrm{Bentham}}$ is not seen.

candidates	n_k	$u_i(y)$	**Bentham**	**Borda**	**Condorcet**	**MV**
uniform	11	$-d^2$	2	1	1	.5
		$-d$	1	1	1	.5
		$-d^{1/2}$	1	1	1	.5
	49	$-d^2$	1.77	1.15	1.04	.62
		$-d$	1.04	1.15	1.04	.62
		$-d^{1/2}$	.83	1.15	1.04	.62
repres.	11	$-d^2$	1.68	.92	.92	.56
		$-d$	.92	.92	.92	.56
		$-d^{1/2}$	.83	.92	.92	.56
	49	$-d^2$	1.89	.93	1.04	.56
		$-d$	1.04	.93	1.04	.56
		$-d^{1/2}$	.85	.93	1.04	.56

Table 1: Various specifications and choices with a Log-Normal Distribution

(100 simulations)	**Bentham**	**Borda**	**Condorcet**	**MV**
mean value	1.56	.87	1.01	.65
standard deviation	.28	.17	.13	.12

Table 2: Robustness of the results for 11 representative candidates and quadratic utility

Tables 2 and 3 provide further precision about this robustness analysis. In this model, Borda and Condorcet are not well distinguished, but the ranking $x_{\text{MV}} < x_{\text{Cond}} < x_{\text{Bentham}}$ appears to be robust. This suggests that the observations that were made in the analytical example of section 3 have some generality.

(100 simulations)	**Bentham-Condor**	**Condor-Borda**	**Borda-MV**
mean value	.55	.14	.22
standard deviation	.29	.22	.19

Table 3: Robustness of the results for 11 representative candidates and quadratic utility

References

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