ECOLE POLYTECHNIQUE
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

CLASSICAL ELECTORAL COMPETITION UNDER APPROVAL VOTING

Jean-François LASLIER
François MANIQUET

October 2009

Cahier n° 2009-41

DEPARTEMENT D'ECONOMIE
Route de Saclay
91128 PALAISEAU CEDEX
(33) 1 69333033
http://www.enseignement.polytechnique.fr/economie/
mailto:chantal.poujouly@polytechnique.edu

Classical electoral competition under approval voting

Jean-François LASLIER
Département d'Economie, Ecole Polytechnique,
91128, Palaiseau, France

François MANIQUET
CORE, Voie du Roman Pays, 34,
1348 Louvain-la-Neuve, Belgique

October 21, 2009

Abstract

We study a Downsian model of electoral competition with an arbitrary number of parties. The voting rule is approval voting. We assume that voters are strategic in the sense of the Leader Rule of Laslier (2009, Jnl. Th. Pol.). We show that if a Condorcet winner policy exists, then there exists an electoral competition equilibrium supporting this policy. Moreover, if the set of policies is one-dimensional and voters have single-peaked preferences, then it is the only electoral competition equilibrium.

1 Introduction

In large societies, collective decisions cannot be taken directly but have to be delegated to professional decision makers. In a democracy, these delegates are typically elected through a competitive mechanism. The simplest expression of such a mechanism is the now standard *Downsian* model of Politics (Downs, 1951) in which a relatively small number of candidates face a relatively large number of voters, the candidates are purely office-motivated and the voters policy-motivated. For the purpose of winning the election, each candidate freely and independently proposes a policy from a fixed and common set of possible policies. Voters are only interested in policies and not in candidates *per se*. They trust that the elected candidate will implement the policy she is proposing.

The usual case in the literature considers only two candidates under plurality voting. Then voters only face a binary choice, so that each voter simply votes for the candidate whose policy she prefers. In that case, competition for office drives the candidates to propose popular policies. In particular, if there exists a policy preferred to any other by a majority of voters — a Condorcet alternative — then both candidates propose this same policy. This statement is even an if and only if statement since, as soon as no Condorcet alternative exists, the two-party Downsian game has no pure-strategy equilibrium. Formal political science has studied this question in great details, and the literature on two-party competition under plurality rule is very large; see for instance the books of Ordeshook (1992), Roemer (2001), Mueller (2003), or Austen-Smith and Banks (2005). This chapter is devoted to the study of multiparty electoral competition under approval voting in the Downsian political context where collective choice is delegated to office-motivated candidates. To recall, approval voting is the electoral rule under which voters are given the right to approve of as many candidates as they wish, and each approval gives one point to the approved candidate. The winner of the election is the candidate having received the largest number of approvals.

Rational behavior of the voters rests on their beliefs about two things. On one hand, they have to wonder which candidate is most likely to win the election and which candidates can challenge this front-runner. On the other hand, they have to make up their mind as to the policies that each candidate would implement, if elected. Rational behavior of the candidates choosing platform campaigns, in turn, rests on their knowledge about the choices of the voters and of the other candidates. We study electoral competition in a framework where the candidates choose rationally (and simultaneously) their political platforms, and the voters react to these platforms. With more than two candidates, not only the voting rule matters but the behavior of voters is no longer as straightforward as it is with two candidates. Studying approval voting, we make the assumption that voters follow the Leader Rule, a behavioral rule which has a rational foundation (Laslier 2009) and satisfies the criterion of sincerity of Brams and Fishburn (1986) and admissibility of Dellis (2010).

We prove that when voters follow that rule, the outcome of the electoral competition among candidates converges towards the Condorcet winner policy in the following sense: if a Condorcet winner policy exists, then there exists an equilibrium that supports it; and if, moreover, the set of policies is one-dimensional and voters' preferences are single-peaked, then this equilibrium is the only one. The prediction of the model is thus that the approval voting electoral rule drives office-motivated candidates to policy moderation.

In Section 2, we present the model and we recall the definition of the Leader Rule. In Section 3, we present the results. In Section 4, we discuss some possible extensions.

2 The model

2.1 Candidates, voters, and preferences

There is a set X of possible policies. We consider two models below. In the first one, X is a finite set with no particular structure. In the second one, X is the real line.

In both models we make the following assumptions on voters and candidates. There is a set $\{1, ..., N\}$ of N voters. Voters have preferences over X. There is a set $\{1, ..., K\}$ of K candidates. Each candidate $k \in \{1, ..., K\}$ has to choose a policy

$$x_k \in X.$$

We assume that policy x_k is implemented if candidate k is elected. Consequently, a voter prefers candidate k to candidate k' if and only if she prefers x_k to $x_{k'}$, $k, k' \in \{1, ..., K\}$, and we can equally well speak in terms of preference over candidates or preference over policies.

The way voters vote among policies $\{x_1, ..., x_K\}$ is described below. Let us begin by describing the preferences of the candidates. As a result of the election, a fraction of the voters, which we denote by $s(k)$, approve of policy x_k, $k \in \{1, ..., K\}$. The number of approvals of k is thus $Ns(k)$. This number is called the *score* of k. The winning candidate is the one with highest score. If several candidates obtain the same, highest, score, the winner is decided by a fair lottery. We assume that the objective of a candidate is to maximize the probability of winning the election.

Let $x, y \in X$. Voters may prefer x to y, y to x, or be indifferent. We assume that the profile of voters' preferences over X is fixed. Given this preference profile of the population $\{1, ..., N\}$, we can compute $g(x, y) \in [0, 1]$, the fraction of the voters who strictly prefer x to y and $i(x, y) \in [0, 1]$, the fraction of the voters who are indifferent between x and y. Note that $g(x, y) - g(y, x)$ measures the relative plurality in favor of policy x against policy y. By definition,

$$g(x, y) + g(y, x) + i(x, y) = 1.$$

We suppose that the number of voters is large.

2.2 Individual voting behavior

Let us now describe how voters choose their vote. Here, we follow the behavioral rule developed in Laslier (2009) and we adapt it to the current model. A rational voter responds to the number of approval votes granted by the other voters to the various candidates (their scores). Let us assume that $s(k)$ represents the fraction of voters approving of k when we do not take account of a given voter's vote. First, the voter deduces from $(s(1), \ldots, s(K))$ a strict ranking $c_1, c_2, ..., c_K$ of the candidates. Candidate c_1 is the leader, according to that voter. This ranking needs to be compatible with the scores in the following sense.

Definition 1 *The ranking $c_1, c_2, ..., c_K$ of the candidates is* ***compatible with a score vector*** $s = (s(1), \ldots, s(K))$ *if for all $k, k\prime \in \{1, \ldots, K\}$,*

$$s(c_k) > s(c_{k'}) \Rightarrow k < k'.$$

If the score vector is such that the K candidates have distinct scores then there is a unique compatible ranking. That is the case analysed in Laslier (2009). Otherwise, the candidates with identical scores can be ranked in any way, providing multiple compatible rankings.

Recall that each voter has fixed preferences over X. For any list of policy positions $x = \{x_1, \ldots, x_K\}$, there is an induced preference relation of this voter over candidates. When the voter has strict preferences over the candidates, the Leader Rule stipulates that she approves of all the candidates she strictly prefers to c_1 and of no candidate she finds strictly worse than c_1, and she votes for c_1 if and only if she prefers c_1 to c_2. When the voter is likely to be indifferent to several candidates, the rule can be generalized as follows.

Assumption 1 ***Leader Rule:*** *Given a strict ranking $c_1, c_2, ..., c_K$ of the candidates, a voter behaves as follows:*

- *If she is indifferent among all candidates, then she approves each of them with probability $1/2$, independently.*

In all other cases, for any candidate d:

- *If the voter is not indifferent between d and c_1, she approves of d if and only if she prefers d to c_1.*
- *If she is indifferent between d and c_1 (for instance in the case $d = c_1$), she approves of d if and only if she prefers d to c_i, where c_i is the first candidate, according to the ranking $c_1, c_2, ..., c_K$ who is not indifferent with c_1.*

If the score vector is such that the K candidates have distinct scores, then this postulated behavior defines a unique ballot for any voter, except in the case where all candidates are indifferent. If the score vector contains ties, several rankings of the candidates are compatible. That may lead to different responses for some voters. For instance, let us assume that $s(1) = s(2) > s(3)$ and the preferences of the voter are: Candidate 1 is preferred to candidate 3, preferred to candidate 2. If the strict ranking of the candidates compatible with scores is 1, 2, 3, then the voter approves only of 1. If the ranking is 2, 1, 3, then she approves of 1 and 2.

Let us briefly present the rationale for the Leader Rule. Assume that the scores represent how voters plan to vote, but for each voter and for each candidate she plans to approve, there is a small chance ϵ that the vote is not recorded, or that she forgets to cast that vote, etc. Then, the actual number of approvals for a candidate $k \in K$ becomes a random variable of mean $(1-\epsilon)Ns(k)$. As a consequence, in spite of the fact that the expected scores of two candidates differ, there is always a positive probability that they tie, so that the vote of this voter is pivotal. Reasoning on these pairwise ties and neglecting three-way ties, a voter votes for a candidate if and only if the most likely serious tie event involving that candidate is one where the former is strictly preferred to the latter (a tie is serious if the voter is not indifferent between the two candidates). Laslier (2009) proves that it gives the above voting behavioral rule, and Nunez (2010) presents this rule and other related models for large electorates.

2.3 Electorate voting

To define the electorate response to a score vector $s = \{s(1), \ldots, s(K)\}$, suppose that s has exactly M compatible rankings. We make the assumption that a proportion $1/M$ of the population of voters adopts each of these rankings, independently of the types. For instance, in the above example $s(1) = s(2) > s(3)$, among all voters sharing the same preferences, fifty per cent will behave according to the ranking 1, 2, 3, and fifty per cent according to 2, 1, 3. This assumption only makes sense in sufficiently large populations. This is precisely our definition of a large population.

Assumption 2 ***Uniform tie-breaking:*** *Given a score vector s, each voter chooses a ranking of the candidates compatible with s, and responds to this ranking. The choice of the ranking is uniform among the rankings compatible with s, and it is independent of the voter's preferences and of the other voters' choices.*

The above assumptions form a simple and natural extension of the Leader Rule defined by Laslier (2009) to handle the possibility of ties, although it is only justified by some kind of law of insufficient reason, as is often the case for uniform rules. Notice that, in any case, since each ballot is defined by the Leader Rule applied to the appropriate ranking, all ballots are sincere and admissible.

2.4 Equilibrium

We define an electoral competition game as one with $K + N$ players, the K candidates and the N voters. In the first stage of the game, each candidate k chooses a policy $x_k \in X$. In the second stage of the game, voters vote, using approval voting. Each vote has a given probability of not being recorded, as explained above. Depending on which votes are recorded, candidates receive numbers of approvals. The candidate with the largest number of approvals is the winner of the election. Ties are broken by a fair lottery. Voters derive utility from the (lottery over) policies that were chosen by the winning candidates. Candidates derive utility from the probability of being elected.

If we restrict our attention to the second stage of the game, then we are back to the game studied by Laslier (2009) except that candidates' score vector can now contain ties. The Leader Rule tells us how voters react to the expected vector of scores. Now, the expected vector of scores itself is completely determined by the voters' expected votes. The equilibrium notion we consider, which we call uniform consistency, is the fixed point of that relation.

Definition 2 *A score vector $s = (s_1, s_2, ..., s_3)$ is* ***uniformly consistent*** *with policy positions $x = (x_1, x_2, ..., x_K)$ if s is the score vector that is obtained when voters react, according to assumptions 1 and 2, to s itself.*

We are interested in the subgame perfect equilibria of the electoral competition game. Using the above definition, such an equilibrium is a pair (x, s) of positions and scores such that s is uniformly consistent with x and for no candidate k there exists a unilateral deviation $x' = (x'_k, x_{-k})$ and a score vector s' uniformly consistent with x' such that the probability of k winning the election is higher in s' than in s.

For the sake of completeness, we prove the following result, which consists of adapting Laslier's result to the current framework. That results concerns cases where candidates choose policies in such a way that no voter is indifferent between any two policies. In those cases, if a strict Condorcet winner policy exists, then there is a unique score vector uniformly consistent with it. That vector is easily built by using the g function.

Recall that a Condorcet winner policy is one that is preferred to any other by a majority of voters.

Definition 3 *A list of policy positions $x = (x_1, x_2, ..., x_K)$ admits a Condorcet winner policy x_ℓ if and only if for all $k \neq \ell$,*

$$g(x_\ell, x_k) \geq 1/2.$$

It admits a strict Condorcet winner if the above inequality is strict for all $k \neq \ell$.

A policy x_ℓ may be a Condorcet winner in a list of policy positions $x = (x_1, x_2, ..., x_K)$ even if there exists a policy $y \in X$ that is preferred to x_ℓ by a majority of voters, provided y is not in the list of policy positions. We will come back on that key issue in the next section.

Proposition 1 *Let us assume that the list of policy positions $x = (x_1, x_2, ..., x_K)$ is such that no voter is indifferent between any pair of policies. If x admits* *a strict Condorcet winner policy x_ℓ, then there exists a unique score vector s that is uniformly consistent with x. This score vector is defined by: For all $k \neq \ell$, $s_k = g(x_k, x_\ell) < 1/2$, and*

$$s_\ell = \min_{k \neq \ell} g(x_\ell, x_k).$$

Proof. 1) The score vector s is consistent with x: Let $\mathcal{L}^{(2)}$ denote the set of candidates obtaining the second largest score. By construction, for all $k, k' \in \mathcal{L}^{(2)} : g(x_k, x_\ell) = g(x_{k'}, x_\ell)$. For all $k \in \mathcal{L}^{(2)}$, a fraction $\frac{1}{|\mathcal{L}^{(2)}|}$ of voters has ranking $\ell, k, \ldots$. Among those voters, a fraction $g(x_\ell, x_k)$ vote for ℓ, a fraction $g(x_k, x_\ell)$ vote for k, and for all $h \neq \ell, k$, a fraction $g(x_h, x_\ell)$ vote for h. Consequently,

$$s(\ell) = \frac{1}{|\mathcal{L}^{(2)}|} \sum_{k \in \mathcal{L}} g(x_\ell, x_k) = g(x_\ell, x_{k'}) \ \forall k' \in \mathcal{L}^{(2)},$$

and for all $k \neq \ell : s(k) = g(x_k, x_\ell)$. Finally, as for all $k \in \mathcal{L}^{(2)}$, $k' \notin \mathcal{L}^{(2)} \cup \{\ell\} : s(k) > s(k')$, by construction, $g(x_k, x_\ell) > g(x_{k'}, x_\ell)$. This implies $g(x_\ell, x_k) < g(x_\ell, x_{k'})$ so that

$$s_\ell = \min_{k \neq \ell} g(x_\ell, x_k).$$

2) Unicity: Let $\mathcal{L}^{(1)}$ denote the set of candidates obtaining the largest score. Assume $\ell \notin \mathcal{L}^{(1)}$. Then,

$$s(\ell) = \frac{1}{|\mathcal{L}^{(1)}|} \sum_{k \in \mathcal{L}^{(1)}} g(x_\ell, x_k) > \frac{1}{2}.$$

First case: $\mathcal{L}^{(1)}$ contains more than one candidate. Then, for all $k \in \mathcal{L}^{(1)}$,

$$\begin{aligned} s(k) &= \frac{1}{|\mathcal{L}^{(1)}|} \sum_{\binom{k' \in \mathcal{L}^{(1)}}{k' \neq k}} g(x_k, x_{k'}) + \frac{1}{|\mathcal{L}^{(1)}|} \frac{1}{|\mathcal{L}^{(1)}| - 1} \sum_{\binom{k' \in \mathcal{L}^{(1)}}{k' \neq k}} g(x_k, x_{k'}) \\ &= \frac{1}{|\mathcal{L}^{(1)}| - 1} \sum_{\binom{k' \in \mathcal{L}^{(1)}}{k' \neq k}} g(x_k, x_{k'}). \end{aligned}$$

Summing up these scores, and recalling that $g(x_k, x_{k'}) + g(x_{k'}, x_k) = 1$ (no voter is indifferent between x_k and $x_{k'}$), we get

$$\sum_{k \in \mathcal{L}^{(1)}} s(k) = \frac{1}{|\mathcal{L}^{(1)}| - 1} \sum_{k,k' \in \mathcal{L}^{(1)}} g(x_k, x_{k'}) + g(x_{k'}, x_k)$$
$$= \frac{1}{|\mathcal{L}^{(1)}| - 1} \frac{(|\mathcal{L}^{(1)}|)(|\mathcal{L}^{(1)}| - 1)}{2} = \frac{|\mathcal{L}^{(1)}|}{2},$$

so that for all $k \in \mathcal{L}^{(1)} : s(k) = \frac{1}{2}$. To summarize, we have $s(k) < s(\ell)$, a contradiction. Second case: $\mathcal{L}$ contains one candidate, say 1. We must have $s(1) > \frac{1}{2}$ and for all k in the set of candidates ranked second, $s(k) = g(x_k, x_1) < \frac{1}{2}$, which is inconsistent with $s(\ell) > \frac{1}{2}$. ■

There are two important directions in which the above result does not extend. First, even if the profile has a Condorcet winner, if the Condorcet winner is not strict, then it is possible that no uniformly consistent scores exist.

Example 1 *Consider a set of three candidates* $\{1, 2, 3\}$ *such that the pairwise comparisons among candidates are:* $g(x_1, x_2) = .5$, $g(x_1, x_3) = .6$, $g(x_2, x_3) = .1$. *No uniformly consistent scores exist for this profile. To see that one can check the impossibility for each ordering, strict or not, of the candidates according to* s. *For instance if* 1 *is alone at the first place in* s, 2 *at the second place, and* 3 *at the third, then the scores should be* $s_1 = g(x_1, x_2) = .5$ *and* $s_2 = g(x_2, x_1) = .5$ *also, a contradiction. If* 1 *and* 2 *tie at the first place and* 3 *comes third, then* $s_1 = g(x_1, x_2) = .5$, $s_2 = g(x_2, x_1) = .5$, *and* $s_3 = (g(x_3, x_1) + g(x_3, x_2))/2 = .65 > .5$, *a contradiction. The reader will easily complete this proof.*

Second, if (a non-negligeable fractions of) voters have indifferences, then there may be several score vectors uniformly consistent with the policy positions and even a strict Condorcet winner may fail to be ranked first in such a score vector.

Example 2 *Consider a set of three candidates* $\{1, 2, 3\}$ *and their policy positions* $x = (x_1, x_2, x_3)$ *inducing the preferences described in the following table (which reads: 4 voters are indifferent between* 1 *and* 2 *and strictly prefer any of these two to* 3, *etc.):*

4	*3*	*2*
1, 2	3	3
3	1	2
	2	1

Observe that 3 *is a strict Condorcet winner. Nevertheless consider the following score vector:* $s(1) = 7, s(2) = 6, s(3) = 5$*, in which candidate* 3 *is ranked last. One can easily check that, applying the Leader Rule, all voters vote for two candidates, and that this score vector is consistent with* x*.*

Of course, if the profile of candidates has no Condorcet winner, it is all-the-most possible that no uniformly consistent score exists. The fact that uniformly consistent scores may fail to exist is a difficulty for the study of electoral competition under AV in the general case, under the uniform tie-breaking assumption. The results (in the next section) will thus be limited to some observations, in the case of existence of a Condorcet winner.

The result conveyed in example 2 above involves preferences that are not single-peaked: the only rankings compatible with the existence of preferences 312 and 321 would have 3 in the middle, which excludes preferences (12)3. The last result in the next section shows that example 2 cannot hold if preferences are single-peaked.

3 Results

We are now equipped to prove our two results. Both results hold even if agents have indifferences among some pairs of policy positions. They confirm the close relationship between approval voting and the Condorcet winner. Proposition 3 essentially states that a Condorcet winner policy, if it exists, can always result from electoral competition under approval voting, and Proposition 3 essentially states that, in single-peaked domains, this is the only possible outcome. The section is completed by showing that these results do not hold if approval voting is replaced with plurality voting.

3.1 Condorcet-consistency

In the previous section, we defined a Condorcet winner by reference to a list $x = (x_1, \ldots, x_K)$ of policy positions. If we look at the entire set X of possible policies, we can define a Condorcet winner policy as one that is preferred to any other policy in X by a majority of voters. Let us note that there is no logical relation between the existence of a Condorcet winner in X and the existence of a Condorcet winner relative to a list of K policy positions in X.

The first result states that if a strict Condorcet winner exists in X, then, independently of the structure of X, all candidates choosing that policy position is an equilibrium.

Proposition 2 *If $x^C \in X$ is a Condorcet winner policy then the strategy profile in which all candidates choose x^C is an equilibrium of the electoral competition game. If x^C is a strict Condorcet winner then the equilibrium is strict.*

Proof. Let $x^C \in X$ be a Condorcet winner policy. Let $x = (x_1, \ldots, x_K)$ be defined by: For all $k \in \{1, \ldots, K\} : x_k = x^C$. Then, by the Leader Rule, each candidate is elected with probability $1/K$. Suppose candidate 1 (for instance) deviates to $x_1 \neq x^C$. There are now two different policy positions to choose from. Independently of how they are ranked, we have $s(1) = g(x_1, x^C) \leq .5$ and for all $k \in \{2, \ldots, K\} : s(k) = g(x^C, x_1) \geq .5$. Candidate 1's probability of being elected is now either $1/K$ or 0. In any case, the deviation is not profitable. If x^C is a strict Condorcet winner policy, then $s(1) = g(x_1, x^C) < .5$ and for all $k \in \{2, \ldots, K\} : s(k) = g(x^C, x_1) > .5$. Consequently, 1's probability of being elected decreases to 0. ■

3.2 Median convergence

We consider in this section the standard, one-dimensional, single-peaked model. The set of possible policies is the real line $X = \mathbb{R}$. Each voter $j \in N$ has a preferred policy p_j. Moreover, for two policies $x, y \in X$ on the left of p_j (resp., on the right of p_j), x is strictly preferred to y if and only if x is closer to p_j than y: $y < x < p_j$ (resp., $p_j < x < y$). Let $x^m \in X$ be the median of the voters' preferred policies — as many voters have their preferred policy at the left as at the right of x^m. We suppose that this point exists and is unique. Then, as is well-known, this policy-moderated, centrist outcome, x^m is a strict Condorcet winner: for any $y \neq x^m$, $g(x^m, y) > .5$, a strict majority of the population strictly prefers x^m to y. The previous result applies and all candidate policy positions being concentrated at the median point is a strict equilibrium of the electoral competition. The following proposition, our main result, also proves it is the only equilibrium. That shows that, under approval voting, electoral competition drives candidates to propose the Condorcet policy platform.

Proposition 3 *In the single-peaked model: (i) The strategy profile in which all candidates choose the median policy position is a strict equilibrium of the electoral competition game. (ii) It is the only equilibrium.*

Proof. Point (i) follows from Proposition 2. (ii) We first note three facts related to the single-peaked profile structure. Let $x, y, z \in X$ be such that $x < y < z$.

- *Fact 1:* no voter is indifferent between the three positions.
- *Fact 2:* the voters (if any) who are indifferent between x and z strictly prefer y to both x and z.
- *Fact 3:* $g(y, x) \geq g(z, x)$ and $g(y, z) \geq g(x, z)$.

Next we observe that, from the definition of the Leader Rule (Assumption 1), if two candidates k, k', propose the same policy $x_k = x_{k'}$, they obtain the same number of votes and any other candidate l obtains the same number of votes as l would obtain if there was only one candidate at position x_k. Now, let $x = (x_1, \ldots, x_K)$ be some list of policy positions chosen by the candidates. For the ease of reading, and when no confusion in the course of the proof can arise, we can neglect the possibility of several candidates located at the same position and we will speak of "a set of candidates" rather than "a set different policy positions chosen by candidates."

There is at least one candidate, say 1, with a probability of winning the election less than or equal to $1/K$. We will prove that deviating to $x'_1 = x^m$ is profitable.[1] Let $s = (s(1), \ldots, s(K))$ be a score vector associated to $x' = (x'_1 = x^m, x_2, \ldots, x_K)$. Notice that $s(1)$ is equal to or larger than some average of $g(x^m, x_k)$, for $k \in \{2, ..., K\}$, and because x^m is a strict Condorcet winner, $s(1) > .5$. Let $\mathcal{L}^{(1)}$ denote the set of candidates obtaining the largest score.

If all the candidates except candidate 1 are located at x^m then the probability of winning goes from 0 to $1/K$ when candidate 1 deviates to x^m. We

[1] We will thus prove that the equilibrium is in dominant strategy.

can thus suppose that some candidates are not located at x^m and we will prove that when deviating to x^m, $1 \in \mathcal{L}^{(1)}$ and $\mathcal{L}^{(1)}$ contains at most $K-1$ candidates, which makes the deviation profitable for candidate 1. Assume $1 \notin \mathcal{L}^{(1)}$. We distinguish three cases.

Case 1: $\mathcal{L}^{(1)}$ contains one candidate, say 2. Let $\mathcal{L}^{(2)}$ denote the set of candidates obtaining the second largest score. Assume that $\mathcal{L}^{(2)}$ contains more than one candidate. By Fact 1, all the scores of 2 and the candidates of $\mathcal{L}^{(2)}$ are determined by the preferences over these candidates. To simplify, let $\mathcal{L}^{(2)} = \{k, k'\}$ (the argument extends if there are more than two candidates). If $x_k < x_2 < x_{k'}$, then

$$\begin{aligned} s(2) &= \frac{g(2,k)+g(2,k')}{2} + i(2,k) + i(2,k'), \\ s(k) &= g(k,2) + i(2,k), \\ s(k') &= g(k',2) + i(2,k'). \end{aligned}$$

We compute $s(k)+s(k') = g(k,2)+g(k',2)+i(2,k)+i(2,k') < 1$, so that $s(k) = s(k') < .5$, in contradiction to the fact that $s(1) > .5$. If $x_2 \notin (x_k, x_{k'})$, then (assuming, w.l.o.g., $x_k \in (x_2, x_{k'})$)

$$\begin{aligned} s(2) &= \frac{g(2,k)+g(2,k')}{2} + i(2,k), \\ s(k) &= g(k,2) + i(2,k), \\ s(k') &= g(k',2). \end{aligned}$$

Given that $g(k,2) \geq g(k',2)$, this implies $g(k,2) = g(k',2)$ and $i(2,k) = 0$. Therefore, $s(2) > s(k)$ implies $s(k) < .5$, in contradiction to the fact that $s(1) > .5$. That proves that $\mathcal{L}^{(2)}$ contains one and only one candidate. We cannot have $\mathcal{L}^{(2)} = \{1\}$, as this would imply $s(2) < s(1)$, a contradiction. Let $\mathcal{L}^{(2)} = \{k\}$. Therefore, we must have $s(2) > s(k) > \ldots \geq s(1) \geq \ldots$. That implies $s(1) \geq g(1,2) > .5$. To have $s(k) > s(1)$, it must be the case that $s(2) = g(2,k) + i(2,k)$ and $s(k) = g(k,2) + i(k,2)$. By Fact 2, $x_1' \notin (x_2, x_k)$. Then, either $x_1' < x_2 < x_3$ or $x_3 < x_2 < x_1'$. As a result, $g(1,k) > g(k,2) + i(2,k)$, a contradiction.

Case 2: $\mathcal{L}^{(1)}$ contains two candidates, say 2 and 3, with $x_2 \leq x_3$. We have $s(1) \geq \frac{g(x_1',x_2)+g(x_1',x_3)}{2} > .5$. Note that all voters who are indifferent between x_2 and x_3 vote exactly in the same way, as they all strictly prefer any policy in (x_2, x_3) to either x_2 or x_3, and they prefer x_2 or x_3 to any position out of

(x_2, x_3). Consequently, either

$$\begin{aligned} s(2) &= g(x_2, x_3) + i(x_2, x_3) \\ s(3) &= g(x_3, x_2) + i(x_2, x_3), \end{aligned}$$

or

$$\begin{aligned} s(2) &= g(x_2, x_3) \\ s(3) &= g(x_3, x_2). \end{aligned}$$

In either case, $g(x_2, x_3) = g(x_3, x_2)$ which implies that $x_2 < x_1' = x^m < x_3$ and $s(1) = g(x_1', x_3) + i(x_2, x_3)$. By Fact 3, $s(1) \geq s(2) = s(3)$, contradicting the assumption on the score vector.

Case 3: $\mathcal{L}^{(1)}$ contains three or more candidates, say $2, 3$ and 4, with $x_2 \leq x_3 \leq x_4$ (a similar argument goes true if $\mathcal{L}^{(1)}$ contains more than three agents). By Assumption 2, $s(2), s(3), s(4)$ are determined as the average between the scores that are compatible to any of the six possible strict rankings of $1, 2, 3$. The scores, for each ranking, are as follows.

	$s(2)$	$s(3)$	$s(4)$
234	$g(2,3) + i(2,3)$	$g(3,2) + i(2,3)$	$g(4,2)$
243	$g(2,4)$	$g(3,2)$	$g(4,2)$
324	$g(2,3) + i(2,3)$	$g(3,2) + i(2,3)$	$g(4,3)$
342	$g(2,3)$	$g(3,4) + i(3,4)$	$g(4,3) + i(3,4)$
423	$g(2,4)$	$g(3,4)$	$g(4,2)$
432	$g(2,4)$	$g(3,4) + i(3,4)$	$g(4,3) + i(3,4)$

Using $g(2,3) \leq g(2,4)$ and $g(4,3) \leq g(4,2)$, we obtain

$$\begin{aligned} s(2) &\leq g(2,4) + \frac{i(2,3)}{3} \\ s(3) &= \frac{g(3,2) + g(3,4)}{2} + \frac{1}{3}(i(2,3) + i(3,4)) \\ s(4) &\leq g(4,2) + \frac{i(3,4)}{3} \end{aligned}$$

Given that $g(2,4) + g(4,2) \leq 1$ and $g(3,2) + g(3,4) \geq 1$, we can only have $s(2) = s(3) = s(4)$ if $i(2,3) = i(2,4) = i(3,4) = 0$ and $g(3,2) + g(3,4) = 1$. Consequently, $s(2) = s(3) = s(4) = .5$ whereas $s(1) > .5$, a contradiction.

That proves that $1 \in \mathcal{L}^{(1)}$. We want to prove that all candidates in $\mathcal{L}^{(1)}$ are located at x^m. Assume two different locations are represented in $\mathcal{L}^{(1)}$: $\mathcal{L}^{(1)} = \{1, k\}$ with $x_k \neq x^m$. Then because x^m is a strict Condorcet winner, $s(1) - s(k) = g(1, k) - g(k, 1) > 0$, a contradiction. Moreover, we know from Case 3 above that a three candidate tie is possible only if all the scores are .5, which is impossible if candidate 1 is one of them. It follows that all winning candidates are at x^m. That completes the proof. ■

3.3 Comparison with Plurality voting

The above result should be contrasted with what happens under other voting rules. Consider Plurality rule and take $K > 2$ (at least 3 candidates). The models of rational voting which are similar to the one used here, such as those of Myerson and Weber (1993), Myerson (2002), or Laslier (2009), provide, as can be easily seen, the following behavior.

Rational behavior for the voter, under the plurality rule is basically to vote for the one she prefers among the two first-ranked candidates. Consider the simple case of the single-peaked model on the real line. Suppose that all candidates are at the median, each one receiving $1/K$ of the votes and having thus the probability $1/K$ of being elected. Then suppose that one candidate, say $k = 1$, moves slightly away from this position to some new position, say $x'_1 = x^m + \varepsilon$ on the right of x^m. This produces a situation in which the electorate is essentially split in two: the left-wing prefers $x_2 = x_3 = ... = x_4 = x^m$ and the right-wing prefers $x_1 = x'_1 = x^m + \varepsilon$.

This potentially gives to the mover almost $1/2$ of the votes while the remaining $(K - 1)$ candidates have to share the remaining votes. Under most reasonable assumption as to voters' behavior, the strength of the split-majority phenomena will be such that candidate 1 will be elected with probability 1. Therefore the situation in which all candidates propose the median is not an equilibrium, exept if voters' beliefs are such that all votes gather on two candidates only. An equilibrium is obtained when only two candidates are located at the median and receive half of the votes while the other candidates receive none. This point (only two parties can survive under Plurality voting), which has been emphasized by Cox (1997) after Duverger (1954) is not valid for approval voting. Further studies on this subject also endogenise the number of candidates running for office: see Dellis and Oak (2006) and Dellis (2010).

4 Extensions

We need to discuss two extensions of the above model. First, we have assumed that candidates maximize their probability of winning the election. Alternatively, we could have assumed that they try to maximize their victory margin (or minimize their defeat margin), that is, for all $k \in \{1, \ldots, K\}$, candidate k maximizes

$$\frac{s(k)}{\max_{k' \in \{1,\ldots,K\} \setminus \{k\}} s(k')}.$$

This assumption is more difficult to justify in the case of approval voting than in the standard two-party plurality case because it is not clear whether one should consider the abolute or relative number of approval votes. Anyway, we conjecture that our three results remain true under this assumption.

The second extension is about the source of uncertainty facing voters. We have assumed that each vote of each voter had a fixed probability of not being recorded. We might have assumed, instead, that each voter had a fixed probability of not going to vote. Under this assumption, there is some correlation between the probability that votes are not recorded. Indeed, if a voter planning to vote for k and k' does not vote, none of her two votes are recorded. Unfortunately, our results do not extend to that case. Nunez (2009) has shown that Laslier (2009)'s result does not hold under this alternative assumption. The same kind of example as the one developed in Nunez (2009) applies in the model we have studied in this chapter.

In conclusion, we have found a new kind of elections in which approval voting leads to electing a Condorcet winner. Compared to Laslier (2009), our results show that if policy positions are endogeneous and follow from candidate competition, then strategic voting based on vote uncertainty leads to the election of the Condorcet winner when it exists. A consequence of electoral competition is that voters may be indifferent between pairs of candidates, a case which was excluded from Laslier's analysis. We showed that indifferences could prevent the general result from holding. Nonetheless, when voters have single-peaked preferences, in spite of possible indifferences, electoral competition leads all candidates to propose the median policy platform.

References

[1] David Austen-Smith and Jeffey S Banks (2005) *Positive Political Theory II: Strategy and Structure*. University of Michigan Press.

[2] Steven J. Brams and Peter C. Fishburn (1983) *Approval Voting* Birkhaüser, second edition: Springer 2007.

[3] Gary W. Cox (1987) *Making Votes Count.* Cambridge University Press.

[4] Arnaud Dellis (2010) "Policy Moderation and Endogenous Candidacy in Approval Voting Elections" in JF Laslier and R Sanver (Eds.) *Handbook of Approval Voting*, Heidelberg: Springer.

[5] Arnaud Dellis and Mandar Oak (2006) Approval voting with endogeneous candidates *Games and Economic Behavior* 54: 47-76.

[6] Anthony Downs (1951) *An Economic Theory of Democracy.*

[7] Maurice Duverger (1954) *Les partis politiques.* Paris: Armand Colin.

[8] Jean-François Laslier (2009) "The Leader rule : a model of strategic approval voting in a large electorate" *Journal of Theoretical Politics* 21: 113-136.

[9] Dennis C. Mueller (2003) Public Choice III. Cambridge University Press.

[10] Roger B. Myerson (2002) "Comparison of scoring rules in Poisson voting games" *Journal of Economic Theory* 103: 219-251.

[11] Roger B. Myerson and Robert J. Weber (1993) "A theory of voting equilibrium" *American Political Science Review* 87: 102-114.

[12] Matias Nunez (2009) "Approval Voting and the Poisson-Myerson Environment" *Journal of Theoretical Politics*, forthcoming.

[13] Matias Nunez (2010) "Approval Voting in large electorates" in JF Laslier and R Sanver (Eds.) *Handbook of Approval Voting*, Heidelberg: Springer.

[14] Peter C. Ordeshook (1992) *A Political Theory Primer*. London: Routledge.

[15] John Roemer (2001) *Political Competition: Theory and Applications.* Harvard University Press.