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## PRUDENT CHOICES AND RATIONALITY

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# PRUDENT CHOICES AND RATIONALITY

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**Abstract:** In this paper, we study the relationships between binary lexicographic composition of rationales (see [Tadenuma, 2002]), prudent choices (see [Houy, 2008a]) and refined prudent choices (see [Houy, 2008b]) in the case of multi-criteria decision making. We show that these relationships are linked to the non-emptiness and rationality properties of these choice functions.

**Key Words :** Prudent choices, multi-criteria decision making.

**Classification JEL:** D0

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# 1 Introduction

In a series of recent articles, two processes have been proposed in order to deal with multi-rationale choices. Let us consider two rationales (or criteria). The first process, introduced in [Tadenuma, 2002] and applied in [Tadenuma, 2005] works as follows. Let us construct the following binary preferences: for any pair of alternatives,  $x$  and  $y$ ,  $x$  is preferred to  $y$  if and only if  $x$  is preferred to  $y$  when considering the first rationale, or,  $y$  is not preferred to  $x$  according to the first rationale and  $x$  is preferred to  $y$  according to the second rationale. Then, the choice function is the maximization function of these composed binary preferences. By construction, since it is the maximization function of binary preferences, this composed choice function satisfies the most common rationality axioms (see [Sen, 1993] and [Sen, 1977] for instance),  $\alpha$  and  $\gamma$ . However, it is known that it is almost never non-empty.

The second process has been axiomatized by [Houy, 2008a] in order to avoid the problem of empty choices. When choosing from a any set, prudent preferences are constructed. A prudent preference is one that contains the first rationale and as many instances of the second rationale as possible with the constraint that the prudent preferences remain acyclic. Since prudent preferences are not unique, we define prudent choices as the set of alternatives that maximize at least one set of prudent preferences. By construction, prudent choices are always non-empty. We also know from [Houy, 2008a] how rational they are (they satisfy  $\gamma$  but only a weak version of  $\alpha$ .) We prove in this article that actually, prudent choices satisfy  $\alpha$  if and only if they are equal to the first process described above and then, if and only if the first process makes empty choices. Said differently, if the first process makes empty choices, we can be sure that implementing the second process will solve the problem of empty choices at the cost of irrational choices. Conversely, if the second process makes irrational choices, implementing the

second process will solve the problem of irrational choices at the cost of empty choices.

A third process has been axiomatized by [Houy, 2008b] in order to refine prudent choices. Since prudent preferences are not unique, we define refined prudent choices as the set of alternatives that maximize all the prudent preferences. It is known that refined prudent choices can make non empty choices. Moreover, they satisfy  $\gamma$  but only a weak version of  $\alpha$  (different from the one satisfied prudent choices), see [Houy, 2008b]. We show in this paper that refined prudent choices are a refinement of prudent choices that always choose, possibly among others, the choices made by the first process of choice. However, we also show that refined prudent choices are different from prudent choices if and only if they make empty choices. Moreover, refined prudent choices are equal to the first process of choice making if and only if they are rational.

In the first section we give the notation and a few lemmas. Main results are given in the second section.

## 2 Notation

Let  $X$  be a finite set of alternatives.  $\mathcal{X}$  is the set of all non-empty subsets of  $X$ ,  $\mathcal{X} = 2^X \setminus \emptyset$ . A choice function on  $X$  is a function  $C : \mathcal{X} \rightarrow 2^X$  such that  $\forall S \in \mathcal{X}, C(S) \subseteq S$ . Let  $\mathcal{C}(X)$  be the set of all choice functions on  $X$ .

Let  $P \subseteq X \times X$  be a binary relation on  $X$ . For any subset  $S$  of  $X$ ,  $P|_S$  is the restriction of  $P$  to  $S$ , *i.e.*  $P|_S = \{(a, b) \in P, a, b \in S\}$ .  $P^t$  is the transitive closure of  $P$  *i.e.*  $\forall a, b \in X, (a, b) \in P^t$  if and only if  $\exists n \in \mathbb{N}, \exists a_1, \dots, a_n \in X$  such that  $\forall i \in \{1, \dots, n-1\}, (a_i, a_{i+1}) \in P, a_1 = a$  and  $a_n = b$ . We say that  $P$  is irreflexive if and only if  $\forall a \in X, (a, a) \notin P$ . We say that  $P$  is asymmetric if and only if  $\forall a, b \in X, (a, b) \in P \Rightarrow (b, a) \notin P$ . We say that  $P$  is acyclic if and only if  $(a, a) \notin P^t$ .<sup>1</sup> An asymmetric binary relation will be

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<sup>1</sup>Notice that, by definition, acyclicity implies asymmetry and asymmetry implies ir-

called a preference relation.

Let  $P_1$  and  $P_2$  be two preference relations on  $X$ . We define  $Q(P_1, P_2)$  by:  $\forall a, b \in X, (a, b) \in Q(P_1, P_2)$  if and only if  $(a, b) \in P_1$  or  $[(b, a) \notin P_1 \text{ and } (a, b) \in P_2]$ .

Let  $(P_1, P_2)$  be an ordered pair of preference relations on  $X$  such that  $P_1$  is acyclic. Let  $S \in \mathcal{X}$ . We say that  $P \subseteq X \times X$  is a prudent composition of  $P_1$  and  $P_2$  on  $X$  if

- $P = P_1 \upharpoonright_S \cup Q$  with  $Q \subseteq P_2 \upharpoonright_S$ ,
- $P$  is acyclic and,
- $\forall Q'$  such that  $Q \subset Q' \subseteq P_2 \upharpoonright_S, P_1 \upharpoonright_S \cup Q'$  is cyclic.

Then, a prudent composition of  $P_1$  and  $P_2$  on  $S$  is a binary relation containing  $P_1 \upharpoonright_S$  and as many elements of  $P_2 \upharpoonright_S$  as possible with the constraint that the prudent composition is not cyclic. We denote by  $\widehat{(P_1, P_2)}(S)$  the set of all prudent compositions of  $P_1$  and  $P_2$  on  $S$ . Notice that by definition,  $\widehat{(P_1, P_2)}(S)$  is non-empty if and only if it is well defined or, said differently, if  $P_1 \upharpoonright_S$  is acyclic.<sup>2</sup>

Let  $P \subseteq X \times X$  be a preference relations on  $X$ . We define  $C_P : \mathcal{X} \rightarrow 2^X$  by

$$\forall S \in \mathcal{X}, C_P(S) = \{a \in S, \forall b \in S, (b, a) \notin P\}.$$

We say that  $P$  rationalizes the choice function  $C_P$ .

Let  $P_1, P_2 \subseteq X \times X$  be two preference relations on  $X$  with  $P_1$  acyclic. We define  $C_{(P_1, P_2)}^\cup$  by

$$\forall S \in \mathcal{X}, a \in C_{(P_1, P_2)}^\cup(S) \Leftrightarrow a \in S \text{ and } \exists P \in \widehat{(P_1, P_2)}(S) \text{ such that}$$

$$\forall b \in S, (b, a) \notin P.$$

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reflexivity.

<sup>2</sup>Obviously, if  $\widehat{(P_1, P_2)}(S) = \{\emptyset\}$ , we still have  $\widehat{(P_1, P_2)}(S) \neq \emptyset$ .

We say that  $(P_1, P_2)$   $\cup$ -prudently rationalizes  $C$ . We define  $C_{(P_1, P_2)}^\cap$  by

$$\forall S \in \mathcal{X}, a \in C_{(P_1, P_2)}^\cap(S) \Leftrightarrow a \in S \text{ and } \forall P \in (\widehat{P_1, P_2})(S),$$

$$\forall b \in S, (b, a) \notin P.$$

We say that  $(P_1, P_2)$   $\cap$ -prudently rationalizes  $C$ .

The three following lemmas are axioms in the litterature. The first imposes that the choice function makes always non-empty choices. The second and third are the usual Contraction Consistency (or Chernoff<sup>3</sup> or  $\alpha$ ) and Expansion Consistency (or  $\gamma$ ) axioms (see [Sen, 1993]).

**AXIOM 1 (NE)**

*Let  $C \in \mathcal{C}(X)$ . The choice function  $C$  satisfies NE if and only if  $\forall S \in \mathcal{X}$ ,  $C(S) \neq \emptyset$ .*

**AXIOM 2 ( $\gamma$ )**

*Let  $C \in \mathcal{C}(X)$ . The choice function  $C$  satisfies  $\gamma$  if and only if  $\forall n \in \mathbb{N}$  and  $\forall S_1, \dots, S_n \in \mathcal{X}$ ,  $a \in \bigcap_{i \in \{1, \dots, n\}} C(S_i)$  implies  $a \in C(\bigcup_{i \in \{1, \dots, n\}} S_i)$ .*

**AXIOM 3 ( $\alpha$ )**

*Let  $C \in \mathcal{C}(X)$ . The choice function  $C$  satisfies  $\alpha$  if and only if  $\forall S, T \in \mathcal{X}$  such that  $S \subseteq T$  and  $\forall a \in S$ ,  $a \in C(T)$  implies  $a \in C(S)$ .*

The following lemma is well known in the litterature since it has been stated in [Blair *et al.*, 1954]. A good reference for these results in [Suzumura, 1983].

**LEMMA 1**

*Let  $C \in \mathcal{C}(X)$ .  $C$  satisfies  $\alpha$  and  $\gamma$  if and only if  $\exists P \subseteq X \times X$  such that  $C = C_P$ .  $C$  satisfies NE,  $\alpha$  and  $\gamma$  if and only if  $\exists P \subseteq X \times X$  such that  $P$  is acyclic and  $C = C_P$ .*

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<sup>3</sup>See [Chernoff, 1954]

The following lemmas characterize the choices made by  $C_{Q(P_1, P_2)}$ ,  $C_{(P_1, P_2)}^{\cup}$  and  $C_{(P_1, P_2)}^{\cap}$ . The first result has been proved in [Houy and Tadenuma, 2008], the second in [Houy, 2008a] and the third in [Houy, 2008b].

**LEMMA 2**

Let  $P_1, P_2$  be two preference relations on  $X$ . Let  $S \in \mathcal{X}$  and  $a \in S$ .  $a \in C_{Q(P_1, P_2)}(S)$  if and only if:

- $\forall b \in S, (b, a) \notin P_1$  and,
- $\forall b \in S$  such that  $(b, a) \in P_2, (a, b) \in P_1$ .

**LEMMA 3**

Let  $P_1, P_2$  be two preference relations on  $X$  with  $P_1$  acyclic. Let  $S \in \mathcal{X}$  and  $a \in S$ .  $a \in C_{(P_1, P_2)}^{\cup}(S)$  if and only if:

- $\forall b \in S, (b, a) \notin P_1$  and,
- $\forall b \in S$  such that  $(b, a) \in P_2, (a, b) \in (Q(P_1, P_2) \upharpoonright_S)^t$ .

**LEMMA 4**

Let  $P_1, P_2$  be two preference relations on  $X$  with  $P_1$  acyclic. Let  $S \in \mathcal{X}$  and  $a \in S$ .  $a \in C_{(P_1, P_2)}^{\cap}(S)$  if and only if:

- $\forall b \in S, (b, a) \notin P_1$  and,
- $\forall b \in S$  such that  $(b, a) \in P_2, (a, b) \in (P_1 \upharpoonright_S)^t$ .

As a simple corollary of the preceding lemmas, we can state that  $C_{(P_1, P_2)}^{\cap}$  is indeed a refinement  $C_{(P_1, P_2)}^{\cup}$  that contains  $C_{Q(P_1, P_2)}$ .

**PROPOSITION 1**

Let  $P_1, P_2$  be two preference relations on  $X$  with  $P_1$  acyclic.  $\forall S \in \mathcal{X}, C_{Q(P_1, P_2)} \subseteq C_{(P_1, P_2)}^{\cap}(S) \subseteq C_{(P_1, P_2)}^{\cup}(S)$ .

### 3 Results

We will now study the non-emptiness and rationality properties of  $C_{Q(P_1, P_2)}$ ,  $C_{(P_1, P_2)}^\cup$  and  $C_{(P_1, P_2)}^\cap$ . Proposition 2 states that obviously,  $C_{(P_1, P_2)}^\cup$  satisfies NE.

**PROPOSITION 2**

*Let  $P_1, P_2 \subseteq X \times X$  be two preference relations with  $P_1$  acyclic.  $C_{(P_1, P_2)}^\cup$  satisfies NE.*

**Proof.** By definition.  $\square$

Proposition states that  $C_{(P_1, P_2)}^\cap$  satisfies NE if and only if  $C_{(P_1, P_2)}^\cap = C_{(P_1, P_2)}^\cup$ . Hence, we cannot refine  $C_{(P_1, P_2)}^\cup$  by  $C_{(P_1, P_2)}^\cap$  without falling on a function that is empty for some choice sets.

**PROPOSITION 3**

*Let  $P_1, P_2$  be two preference relations on  $X$  with  $P_1$  acyclic.  $C_{(P_1, P_2)}^\cap = C_{(P_1, P_2)}^\cup$  if and only if  $C_{(P_1, P_2)}^\cap$  satisfies NE.*

**Proof.** If: Assume that  $C_{(P_1, P_2)}^\cap \neq C_{(P_1, P_2)}^\cup$ . Let  $S \in \mathcal{X}$  be such that  $C_{(P_1, P_2)}^\cap(S) \neq C_{(P_1, P_2)}^\cup(S)$ . By Proposition 1,  $C_{(P_1, P_2)}^\cap(S) \subseteq C_{(P_1, P_2)}^\cup(S)$ . Hence,  $\exists a \in C_{(P_1, P_2)}^\cup(S)$  such that  $a \notin C_{(P_1, P_2)}^\cap(S)$ . By Lemma 3,  $a \in C_{(P_1, P_2)}^\cup(S)$  implies  $\forall b \in S, (b, a) \notin P_1$ . Moreover, by Lemmas 3 and 4,  $\exists c \in S, (c, a) \in P_2, (a, c) \in (Q(P_1, P_2) \mid_S)^t, (a, c) \notin (P_1 \mid_S)^t$ . Then,  $(Q(P_1, P_2))$  is cyclic. Let us have  $A$  be the smallest cycle of  $Q(P_1, P_2)$  containing  $a$  and  $c$ . It is straightforward to check that we can denote  $A = \{a_1, \dots, a_{\#A}\}$  with  $(Q(P_1, P_2) \mid_A = \{(a_{\#A}, a_1)\} \cup_{i \in \{1, \dots, \#A-1\}} \{(a_i, a_{i+1})\}$  and with no loss of generality  $c = a_{\#A}, a_1 = a$ . Since  $(a, c) \notin (P_1 \mid_S)^t, \exists i \in \{1, \dots, \#A-1\}$  such that  $(a_i, a_{i+1}) \in P_2$  and  $(a_i, a_{i+1}), (a_{i+1}, a_i) \notin P_1$ . Hence, it is easy to check that  $C_{(P_1, P_2)}^\cap(A) = \emptyset$ .

Only If: By Proposition 2.  $\square$

Proposition 4 gives the conditions under which the predicates of Proposition 3 apply.



**PROPOSITION 4**

Let  $P_1, P_2 \subseteq X \times X$  be two preference relations with  $P_1$  acyclic.  $C_{(P_1, P_2)}^\cap$  satisfies NE if and only if  $\forall A \in \mathcal{X}$ ,  $Q((P_1 \upharpoonright_A)^t, P_2 \upharpoonright_A)$  is acyclic.

**Proof.** If: Let  $A \in \mathcal{X}$ . If  $Q((P_1 \upharpoonright_A)^t, P_2 \upharpoonright_A)$  is acyclic, then it has a maximal element. Let us have  $a \in A$  such that  $\forall b \in A, (b, a) \notin Q((P_1 \upharpoonright_A)^t, P_2 \upharpoonright_A)$ . Then, by definition,  $\forall b \in A, (b, a) \notin P_1$ . Moreover,  $\forall b \in A$  such that  $(b, a) \in P_2$ , by definition of  $Q((P_1 \upharpoonright_A)^t, P_2 \upharpoonright_A)$  we have  $(a, b) \in (P_1 \upharpoonright_A)^t$ . Hence, by Lemma 4,  $a \in C_{(P_1, P_2)}^\cap(A)$ .

Only If: On the contrary, assume  $\exists A \in \mathcal{X}$ , such that  $Q((P_1 \upharpoonright_A)^t, P_2 \upharpoonright_A)$  is cyclic. By definition of  $Q((P_1 \upharpoonright_A)^t, P_2 \upharpoonright_A)$ , it is asymmetric. Then we can define  $n \in \mathbb{N} \setminus \{1, 2\}$  and  $B = \{a_1, \dots, a_n\} \subseteq A$  such that  $\forall i \in \{1, \dots, n\}$ ,  $(a_{i+1}, a_i) \in Q((P_1 \upharpoonright_A)^t, P_2 \upharpoonright_A)$  where we define, for the sake of simplicity,  $a_{n+1} = a_1$ . Let us have  $E = \{a_i \in B, (a_{i+1}, a_i) \in (P_1 \upharpoonright_A)^t\}$ . For all  $a_i \in E$ , define  $n_i \in \mathbb{N} \setminus \{1\}$  and  $D_i = \{d_1, \dots, d_{n_i}\} \subseteq A$  such that  $\forall j \in \{1, \dots, n_i - 1\}$ ,  $(d_{j+1}, d_j) \in P_1 \upharpoonright_A$  with  $d_{n_i} = a_{i+1}$  and  $d_1 = a_i$ . Let us compute  $C_{(P_1, P_2)}^\cap(B \cup_{a_i \in E} D_i)$ . Obviously, for all  $D_i$ ,  $d_{n_i}$  is the only element non dominated by  $P_1$ . Hence, if  $C_{(P_1, P_2)}^\cap(B \cup_{a_i \in E} D_i) \neq \emptyset$ , then,  $C_{(P_1, P_2)}^\cap(B \cup_{a_i \in E} D_i) \subseteq B \setminus E$ . Let us have  $a_k \in B \setminus E$ . By definition,  $(a_{k+1}, a_k) \in P_2$ . Assume  $(a_k, a_{k+1}) \in (P_1 \upharpoonright_{B \cup_{a_i \in E} D_i})^t$ . Then,  $(a_k, a_{k+1}) \in (P_1 \upharpoonright_A)^t$  which contradicts  $(a_{k+1}, a_k) \in Q((P_1 \upharpoonright_A)^t, P_2 \upharpoonright_A)$ . Hence, by Lemma 4,  $a_k \notin C_{(P_1, P_2)}^\cap(B \cup_{a_i \in E} D_i)$ . Hence,  $C_{(P_1, P_2)}^\cap(B \cup_{a_i \in E} D_i) = \emptyset$ .  $\square$

The first result concerning rationality states that obviously, by definition and Lemma 1,  $C_{Q(P_1, P_2)}$  is rational in the sense that it satisfies both  $\alpha$  and  $\gamma$ . Moreover, it has been proved in [Houy, 2008a] and [Houy, 2008b] respectively that  $C_{(P_1, P_2)}^\cup$  and  $C_{(P_1, P_2)}^\cap$  satisfy  $\gamma$ .

**PROPOSITION 5**

Let  $P_1, P_2 \subseteq X \times X$  be two preference relations.  $C_{Q(P_1, P_2)}$  satisfies  $\alpha$  and  $\gamma$ .  $C_{(P_1, P_2)}^\cup$  and  $C_{(P_1, P_2)}^\cap$  satisfy  $\gamma$ .

Proposition states that  $C_{(P_1, P_2)}^\cap$  satisfies  $\alpha$  if and only if  $C_{(P_1, P_2)}^\cap = C_{Q(P_1, P_2)}$ . Hence, we cannot expand  $C_{Q(P_1, P_2)}$  by  $C_{(P_1, P_2)}^\cap$  without falling on a function that is not rational.

**PROPOSITION 6**

Let  $P_1, P_2$  be two preference relations on  $X$  with  $P_1$  acyclic.  $C_{(P_1, P_2)}^\cap = C_{Q(P_1, P_2)}$  if and only if  $C_{(P_1, P_2)}^\cap$  satisfies  $\alpha$ .

**Proof.** If: Assume that  $C_{(P_1, P_2)}^\cap \neq C_{Q(P_1, P_2)}$ . Let  $S \in \mathcal{X}$  be such that  $C_{(P_1, P_2)}^\cap(S) \neq C_{Q(P_1, P_2)}(S)$ . By Proposition 1,  $C_{Q(P_1, P_2)}(S) \subseteq C_{(P_1, P_2)}^\cap(S)$ . Hence,  $\exists a \in C_{(P_1, P_2)}^\cap(S)$  such that  $a \notin C_{Q(P_1, P_2)}(S)$ . Then, by Lemmas 2 and 4,  $\forall b \in S, (b, a) \notin P_1, \exists d \in S, (d, a) \in P_2, (a, d) \notin P_1$  and  $\exists n \in \mathbb{N} \setminus \{1\}$  and  $\exists a_1, \dots, a_n \in S$  such that  $\forall i \in \{1, \dots, n-1\}, (a_i, a_{i+1}) \in P_1, a_1 = a$  and  $a_n = d$ . Then, by definition,  $a \notin C_{(P_1, P_2)}^\cap(\{a, d\})$  and  $a \in C_{(P_1, P_2)}^\cap(\{a_1, \dots, a_n\})$ , contradicting the fact that  $C_{(P_1, P_2)}^\cap$  satisfies  $\alpha$ .

Only If: By Proposition 5.  $\square$

Proposition 7 gives the conditions under which the predicates of Proposition 6 apply.

**PROPOSITION 7**

Let  $P_1, P_2 \subseteq X \times X$  be two preference relations with  $P_1$  acyclic.  $C_{(P_1, P_2)}^\cap$  satisfies  $\alpha$  if and only if  $Q(P_1, P_2) \cup P_1^t$  is asymmetric.

**Proof.** If: Let  $A, B \in \mathcal{X}$  be such that  $A \subseteq B$ . Let  $b \in C_{(P_1, P_2)}^\cap(B)$ . By Lemma 4,  $\forall c \in B, (c, b) \notin P_1$ . Now assume that  $\exists d \in B, (d, b) \in P_2$ . Then, by Lemma 4,  $(b, d) \in P_1^t$ . Hence,  $(d, b) \in P_1$  (else,  $Q(P_1, P_2) \cup P_1^t$  is symmetric contradicting the assumptions). Then,  $\forall d \in B, (d, b) \in P_2 \Rightarrow (b, d) \in P_1$ . Then,  $\forall d \in A, (d, b) \notin P_1$  and  $[(d, b) \in P_2 \Rightarrow (b, d) \in P_1]$ . Hence, by Lemma 4,  $b \in C_{(P_1, P_2)}^\cap(A)$ .

Only If: Let us have  $Q(P_1, P_2) \cup P_1^t$  symmetric. Then,  $\exists a, b \in X$  such that  $(a, b), (b, a) \in Q(P_1, P_2) \cup P_1^t$ . By definition,  $Q(P_1, P_2)$  is asymmetric and  $P_1^t$  is asymmetric since  $P_1$  is acyclic. Hence, let us have, with no loss of

generality,  $(a, b) \in Q(P_1, P_2) \setminus P_1^t$  and  $(b, a) \in P_1^t \setminus Q(P_1, P_2)$ . By definition,  $b \notin C_{(P_1, P_2)}^\cap(\{a, b\})$ . Now, let us have  $n \in \mathbb{N} \setminus \{1, 2\}$  and  $A = \{a_1, \dots, a_n\}$  such that  $\forall i \in \{1, \dots, n-1\}$ ,  $(a_i, a_{i+1}) \in P_1$ ,  $a_1 = b$ ,  $a_n = a$ . Then, by definition,  $\forall c \in A$ ,  $(b, c) \in P_1^t$  and  $\forall c \in A$ ,  $(c, b) \notin P_1$  by acyclicity of  $P_1$ . Hence, by Lemma 4,  $b \in C_{(P_1, P_2)}^\cap(A)$  with  $a \in A$ . Hence a contradiction with  $\alpha$ .  $\square$

Finally, we prove that  $C_{(P_1, P_2)}^\cup$  satisfies  $\alpha$  if and only if  $C_{Q(P_1, P_2)}$  satisfies NE if and only if  $C_{(P_1, P_2)}^\cup = C_{Q(P_1, P_2)}$  and hence, by the following results,  $C_{(P_1, P_2)}^\cap = C_{(P_1, P_2)}^\cup = C_{Q(P_1, P_2)}$ .

**PROPOSITION 8**

Let  $P_1, P_2 \subseteq X \times X$  be two preference relations with  $P_1$  acyclic. The following are equivalent:

1.  $C_{(P_1, P_2)}^\cup$  satisfies  $\alpha$ ,
2.  $C_{(P_1, P_2)}^\cup = C_{Q(P_1, P_2)}$ ,
3.  $C_{(P_1, P_2)}^\cap$  satisfies  $\alpha$  and NE,
4.  $C_{Q(P_1, P_2)}$  satisfies NE,
5.  $Q(P_1, P_2)$  is acyclic.

**Proof.** 5  $\Leftrightarrow$  4: By Lemma 1.

2  $\Leftrightarrow$  3: By Propositions 1, 3 and 6.

3  $\Rightarrow$  1: By Proposition 3, if  $C_{(P_1, P_2)}^\cap$  satisfies NE, then,  $C_{(P_1, P_2)}^\cap = C_{(P_1, P_2)}^\cup$ . Then, if  $C_{(P_1, P_2)}^\cap$  satisfies  $\alpha$ ,  $C_{(P_1, P_2)}^\cup$  does as well.

1  $\Rightarrow$  5: Assume that  $Q(P_1, P_2)$  is cyclic. Since by definition,  $Q(P_1, P_2)$  is asymmetric,  $\exists n \in \mathbb{N} \setminus \{1, 2\}$  and  $\exists a_1, \dots, a_n \in X$  such that  $\forall i \in \{1, \dots, n\}$ ,  $(a_i, a_{i+1}) \in Q(P_1, P_2)$  with  $a_{n+1} = a_1$ . Since  $P_1$  is acyclic, let us set with no loss of generality,  $(a_1, a_2) \in P_2$ ,  $(a_1, a_2), (a_2, a_1) \notin P_1$ . Then, by Lemma 3,  $a_2 \notin C_{(P_1, P_2)}^\cup(\{a_1, a_2\})$  whereas  $a_2 \in C_{(P_1, P_2)}^\cup(\{a_1, \dots, a_n\})$  contradicting  $\alpha$  for  $C_{(P_1, P_2)}^\cup$ .

(5 and 4)  $\Rightarrow$  3: By Proposition 1,  $C_{Q(P_1, P_2)}$  satisfies NE implies that  $C_{(P_1, P_2)}^\cap$  satisfies NE. Let us have  $A, B \in \mathcal{X}$  be such that  $A \subseteq B$ . Let  $a \in C_{(P_1, P_2)}^\cap(B)$ . By Lemma 4,  $\forall b \in B, (b, a) \notin P_1$ . Moreover, by the fact that  $Q(P_1, P_2)$  is acyclic,  $\forall b \in B, (b, a) \notin Q(P_1, P_2)$ . Then,  $\forall b \in A, (b, a) \notin Q(P_1, P_2)$ . Hence, by Lemma 4,  $a \in C_{(P_1, P_2)}^\cap(A)$ .  $\square$

The following examples show that the conditions given in Propositions 4 and 7 are independent. More precisely, let  $P_1$  and  $P_2$  be two preference relations with  $P_1$  acyclic. It not necessarily true that if  $C_{(P_1, P_2)}^\cap$  satisfies NE (resp.  $\alpha$ ), then it satisfies  $\alpha$  (resp. NE).

**EXAMPLE 1**

Let  $X = \{a, b, c, d\}$  and let  $P_1 = \{(a, b), (c, d)\}$  and  $P_2 = \{(b, c), (d, a)\}$ . By Proposition 7, since  $Q(P_1, P_2) \cup P_1^t = \{(a, b), (c, d), (b, c), (d, a)\}$ ,  $C_{(P_1, P_2)}^\cap$  satisfies  $\alpha$ . However,  $C_{(P_1, P_2)}^\cap(X) = \emptyset$  since  $\widehat{(P_1, P_2)}(X) = \{\{(a, b), (b, c), (c, d)\}, \{(a, b), (d, a), (c, d)\}\}$

**EXAMPLE 2**

Let  $X = \{a, b, c\}$  and let  $P_1 = \{(a, b), (b, c)\}$  and  $P_2 = \{(c, a)\}$ . By Proposition 4,  $C_{(P_1, P_2)}^\cap$  satisfies NE. However, it does not satisfy  $\alpha$  since  $C_{(P_1, P_2)}^\cap(X) = \{a\}$  whereas  $C_{(P_1, P_2)}^\cap(\{a, c\}) = \{c\}$ .

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