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Abstract

We examine the conditions under which iterative elimination of weakly dominated strategies refines the set of proper outcomes of a normal-form game. We say that the proper inclusion holds in terms of outcome if the set of outcomes of all proper equilibria in the reduced game is included in the set of all proper outcomes of the original game. We show by examples that neither dominance solvability nor the transference of decision-maker indifference condition (TDI^* of Marx and Swinkels [1997]) implies proper inclusion. When both dominance solvablility and the TDI^* condition are satisfied, a positive result arises: the game has a unique stable outcome. Hence, the proper inclusion is guaranteed.

KEYWORDS: Weak dominance, Iterated elimination, Proper equilibrium.

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1 Introduction

Iterative elimination of weakly dominated strategies ($IEWDS$) in normal-form games is a rather controversial procedure. The main criticism against such a procedure is, perhaps, the most basic one: this procedure is order-dependent. Different strategy profiles might survive through this procedure. For instance, removing one weakly dominated strategy at a time rather than all of them simultaneously might lead to different solutions of a game. However, there are games in which the procedure becomes order-independent. The TDI^* (transference of decision-maker indifference) condition proposed by Marx and Swinkels [1997] ensures that this is the case for normal-form games. Roughly speaking, this condition implies that if a player is indifferent between two pure strategy profiles that differ solely on his action, then the rest of the players are also indifferent. The class of games satisfying this condition is broad, including examples such as first-price auctions, oligopoly games, and patent races.

Independently of these conditions, there are several scholars who have argued that the procedure has an interest *per se*. For example, Farquharson [1969] suggests the sophisticated voting principle according to which a reasonable equilibrium must survive $IEWDS$. Within the more general framework of finite games, Mertens stability (Mertens [1989]) is the equilibrium concept that satisfies the most comprehensive list of desirable game-theoretical properties. Note that, again, stability against iterated elimination of dominated strategies is one of the axioms.

Our work builds a bridge between these two strands of the literature, of which we provide a summary in the next subsection.

1.1 Overview of the results

As previously discussed, the set of strategies which remain after elimination of weakly dominated strategies depends on the order of elimination. Consider, for instance, the following game with $x > 0$ and $y > 0$:

	L	R
T	2,1	x,1
B	2,y	0,0

Depending on the order, the set of remaining strategies is either $\{T\}\times\{L\}, \{T,B\}\times\{L\}$ or $\{T\}\times\{L,R\}$. However, as remarked by Marx and Swinkels [1997], the order is irrelevant as long as the game satisfies the TDI^* (transferrence of decision-maker indifference[1]) condition. In this game, TDI^* is satisfied if and only if $(x,y) = (2,1)$. For such values of x and y, all fully-reduced games are equivalent in terms of

[1] For a formal definition, see Section 2.

payoffs. We ask whether an analogous statement can be made for the set of perfect/proper equilibrium payoffs. Indeed, (T, L) is the unique undominated strategy profile, hence it is the unique perfect and proper equilibrium of the game for any $x > 0$ and $y > 0$. However, removing a weakly dominated strategy (either B or R) enlarges the set of perfect/proper equilibria. Moreover, the payoff of all perfect/proper equilibria does not depend on the order by which the weakly dominated strategies are removed, if and only if the TDI^* condition is satisfied.

Order dependence of the set of perfect/proper equilibria has been studied in the literature (see, for instance, Borm [1992] or Myerson [1997]). However, they focus on the equilibrium strategies rather than on the equilibrium payoffs as we do. Moreover, in their examples, there is a unique perfect/proper equilibrium payoff, and thus the payoff becomes trivially order-independent. Therefore, the first question that we address is the following:

Question A. For the games which satisfy the TDI^* condition, is the set of perfect/proper equilibrium *payoffs* unaffected by the elimination of weakly dominated strategies?

Since the TDI^* condition imposes indifferences in terms of the payoffs, it may be a good candidate to guarantee payoff invariance. However, the answer to Question A is negative. We provide two examples, in Section 3.2 for perfect equilibrium and in 4.1 for proper equilibrium.

The main reason behind the negative answer seems to be related to the existence of connected components of equilibria with a continuum of outcomes. Examples of such components can be found in Govindan and McLennan [2001] and Kukushkin et al. [2008].[2] We slightly modify the previously mentioned examples in order to prove that elimination of weakly dominated strategies might *enlarge* the set of perfect and proper equilibria, even in terms of *outcomes*.

We then move to the games in which iterated elimination of weakly dominated strategies singles out a unique strategy profile: dominance-solvable games. As above, we know from the literature that the surviving profile needs not be a proper equilibrium of the game. However, here we are concerned with the set of *payoffs*. As far as our knowledge goes, the outcome of the solution induced by the dominance-solvability is included (and often coincides with) the outcome of all proper equilibria in terms of the payoffs in the examples given in the literature. Therefore, the second question we address is the following:

Question B. For the dominance-solvable games, does the surviving payoff coincide with the one associated to a perfect/proper equilibrium?

The answer turns out to be negative again, as shown in the example in Section 4.2. This happens because $IEWDS$ and properness choose different profiles in the

[2] See also Pimienta [2010], which proves that such components do not exist in three-outcome bimatrix games.

Nash component. When the Nash component includes a continuum of outcomes as in the example, $IEWDS$ and properness need not induce the same outcome. Note that the novelty of the example is that the surviving *payoffs* obtained by any order of elimination are different from the proper payoffs.

Given the answers to Questions A and B, we are ready to present our main result:

Claim. For the dominance-solvable games which satisfy the TDI^* condition, the surviving payoffs coincide with the unique stable payoffs, hence with those associated to a proper equilibrium.

The work is structured as follows. Section 2 introduces the canonical framework in which we work. Section 3 presents the results dealing with perfection, and Section 4 is focused on the relationship between properness and $IEWDS$.

2 The setting

Let Γ be an n-person normal-form game $\Gamma = (S_1, \ldots, S_n; U_1, \ldots, U_n; N)$, where $N = \{1, 2, \ldots, n\}$ is the set of players, each S_i is a non-empty finite set of pure strategies, and each U_i is a real-valued utility function defined on the domain $S = \prod_{i \in N} S_i$.

For any finite set M, let $\Delta(M)$ be the set of all probability distributions over M. Thus, $\Delta(S_i)$ is the set of mixed strategies for player i in Γ. Similarly, $\Delta^0(S_i)$ stands for the set of *completely* mixed strategies for player i in $\Delta(S_i)$. Furthermore, for any mixed strategy σ_i, its support is denoted by $\text{Supp}(\sigma_i) = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$.

The utility functions are extended to mixed strategies in the usual way:

$$U_j(\sigma_1, \ldots, \sigma_n) = \sum_{(s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n} \left(\prod_{i=1}^{n} \sigma_i(s_i) \right) U_j(s_1, \ldots, s_n).$$

The pure strategy s_j^* is a best response to σ_{-j} for player j iff

$$U_j(s_j^*, \sigma_{-j}) = \max_{s_j' \in S_j} U_j(s_j', \sigma_{-j}).$$

An ε-perfect equilibrium of a normal-form game is a completely mixed strategy profile, such that whenever some pure strategy s_i is a worse reply than some other pure strategy t_i, the weight on s_i is smaller than ε. A perfect equilibrium of a normal-form game is a limit of ε-perfect equilibria as $\varepsilon \to 0$.

An ε-proper equilibrium of a normal-form game is a completely mixed strategy profile, such that whenever some pure strategy s_i is a worse reply than some other pure strategy t_i, the weight on s_i is smaller than ε times the weight on t_i. A proper equilibrium of a normal-form game is a limit of ε-proper equilibria as $\varepsilon \to 0$.

For $W \subseteq S$, let the strategies in W that belong to i be denoted $W_i = W \cap S_i$. Say that $W \subseteq S$ is a *restriction* of S if $\forall i$, $W_i \neq \emptyset$. Note that any restriction W of S generates a unique game given by strategy spaces W_i and the restriction of U_i to $\prod_{i=1}^{n} W_i$.

Definition 1. [Weak Dominance] *Let $\sigma_i, \tau_i \in \Delta(S_i)$ and let W be a restriction of S. Then,*
(i) σ_i very weakly dominates τ_i on W if $U_i(\sigma_i, \gamma_{-i}) \geq U_i(\tau_i, \gamma_{-i}) \forall \gamma_{-i} \in W_{-i} = \prod_{j \neq i} W_j$, and
(ii) σ_i weakly dominates τ_i on W if σ_i very weakly dominates τ_i on W, and, in addition, $U_i(\sigma_i, \gamma'_{-i}) > U_i(\tau_i, \gamma'_{-i})$ for some $\gamma'_{-i} \in W_{-i}$. We write $\sigma_i \succ_W \tau_i$.

Definition 2. [Redundancy] *Let $\sigma_i, \tau_i \in \Delta(S_i)$. Then σ_i is redundant to τ_i on restriction W if for all $\gamma_i \in W_{-i}$, $U(\sigma_i, \gamma_{-i}) = U(\tau_i, \gamma_{-i})$. A strategy τ_i is redundant on W if there is $\sigma_i \in W$ redundant to τ_i.*

Following Marx and Swinkels [1997], we define nice weak dominance, and the TDI and TDI^* conditions.

Definition 3. [Nice Weak Dominance]. *Let $\sigma_i, \tau_i \in \Delta(S_i)$. σ_i nicely weakly dominates τ_i on restriction W if σ_i weakly dominates τ_i on W and for all $\gamma_{-i} \in W_{-i}$, $U_i(\sigma_i, \gamma_{-i}) = U_i(\tau_i, \gamma_{-i})$ implies $U(\sigma_i, \gamma_{-i}) = U(\tau_i, \gamma_{-i})$.*[3]

Definition 4. [TDI]. *Game Γ satisfies TDI if $\forall i \in N, \forall s_i, r_i \in S_i, U_i(s_i, s_{-i}) = U_i(r_i, s_{-i}) \Longrightarrow U_j(s_i, s_{-i}) = U_j(r_i, s_{-i})$.*

In a game which satisfies TDI, if an agent is indifferent between two pure-strategy profiles which differ only on her action, then the indifference is transferred to the other agents as well. Marx and Swinkels [1997] show that if a game satisfies TDI, then it generically satisfies TDI^*.

Definition 5. [TDI^*]. *Game Γ satisfies TDI^* if for all restrictions W, $\forall i \in N$, and $\forall s_i \in S_i$, if s_i is very weakly dominated on W by $\sigma_i \in \Delta(S_i \setminus s_i)$, then $\exists \sigma'_i \in \Delta(S_i \setminus s_i)$ such that either s_i is weakly dominated on W by σ'_i or s_i is redundant on W to σ'_i.*

If a game satisfies TDI^*, then whenever player i is indifferent between strategies s_i and σ_i, fixing the profile of opponents' strategies s_{-i}, either all players are indifferent between profiles (s_i, s_{-i}) and (σ_i, s_{-i}), or there is some strategy σ'_i such that i strictly prefers σ'_i over s_i and σ_i given s_{-i}.

Remark: For games satisfying TDI^*, weak dominance is equivalent to nice weak dominance.

[3]This notion is defined as Nice Weak Dominance* in Marx and Swinkels [1997]. However, we omit * in this paper for simplicity.

Definition 6. [Reduction] *Let V be a restriction of S, and let W be a restriction of V. Then W is a reduction of V by weak dominance if $W = V \setminus \{X^1,...,X^m\}$, where $\forall k$, $X^k \subset S$, and $\forall x \in X^k$, $\exists z \in V \setminus \{X^1,...,X^k\}$ such that z weakly dominates x on $V \setminus \{X^1,...,X^{k-1}\}$. W is a full reduction of V by weak dominance if W is a reduction of V by weak dominance and no strategies in W are weakly dominated on W.*

In other words, a reduction is the result of iteratively removing sets of strategies that are weakly dominated. A full reduction is one in which no weakly dominated strategies are left. Note that the same definition applies if replacing weak dominance by nice weak dominance or very weak dominance.

Let Γ^k denote the reduced game after k rounds of successive reductions, and let $S_i^k \subseteq S_i^{k-1}$, $S^k \subseteq S^{k-1}$ be the corresponding strategy spaces. We write $S^0 = S$ and $\lim_{k\to\infty} S^k = \cap_{k=0}^{\infty} S^k = S^\infty$. Γ^∞ denotes the reduced game with strategy space S^∞ and the restriction of U_i to S^∞.

Definition 7. [Dominance Solvability]. *The game Γ is dominance-solvable if there exists a sequence $\Gamma^0, \Gamma^1, \ldots, \Gamma^\infty$ such that:*

(a) *Γ^{k+1} is a reduction from Γ^k,*

(b) *$\Gamma^0 = \Gamma$,*

(c) *in Γ^∞, each player has exactly one pure strategy.*

3 Perfect equilibria

For any game $\Gamma = (S, U)$, let $Pe(\Gamma)$ denote its set of perfect equilibria and $Pro(\Gamma)$ denote its set of proper equilibria. The sets of (Nash) equilibria and undominated equilibria of Γ are respectively denoted $Ne(\Gamma)$ and $UNe(\Gamma)$.

By iterated weak dominance, there exists a finite number of orders (as there is a finite number of strategies, and we assume that at least one strategy is deleted at each stage until the game is fully reduced). Let Θ be the set of all possible orders of reduction. Successive reductions of a game Γ due to order $o \in \Theta$ are as follows:

$$\Gamma_o^0 = \Gamma = (S, U),\ \Gamma_o^1 = (S_o^1, U),\ \Gamma_o^2 = (S_o^2, U), \ldots,\ \Gamma_o^\infty = (S_o^\infty, U)$$

with $S_o^i \supseteq S_o^{i+1}$.

Γ_o^∞ stands for the fully reduced game obtained through iterated weak dominance by the order of reduction o.

It is simple to understand that the set of perfect equilibria of a reduced game is not nested in the whole set of perfect equilibria. The next well-known example proves that removing either M, C or both M and C leads to different sets of perfect equilibria for the reduced games, whereas the unique perfect equilibrium of the whole game is (T, L).

	L	C
T	2,1	1,1
M	2,1	0,0

However, despite this path-dependent procedure, we can state the following result.

Proposition 1. *For any order of reduction $o \in \Theta$, $Pe(\Gamma_o^k) \cap Pe(\Gamma) \neq \emptyset \ \forall k \geq 1$.*

Proof. We omit the definition of Mertens stable sets (Mertens [1989] for a complete definition). We simply use three of its properties. First, stable sets always exist. Second, stable sets are connected sets of normal-form perfect equilibria (connectedness). Third, stable sets of a game contain stable sets of any game obtained by deleting a pure strategy which is at its minimum probability in any normal form ε-perfect equilibrium in the neighborhood of the stable set (iterated dominance and forward induction). Hence, the last property applies in particular to any weakly dominated strategy. Therefore, there exists at least one stable set of Γ_o^k which is included in a stable set of Γ_o^{k-1}. As any point in a stable set is a perfect equilibrium, we can directly conclude. □

We can therefore state the next corollary without proof.

Corollary 1. *For any order of reduction $o \in \Theta$, $Pe(\Gamma_o^\infty) \cap Pe(\Gamma) \neq \emptyset$.*

3.1 Bimatrix games

Within the set Θ, let m stand for the maximal simultaneous reduction by weak dominance in which all mixed and pure strategies that are weakly dominated by some (mixed) strategy are removed at each step.

Proposition 2. *Let Γ be a bimatrix game. By maximal simultaneous reduction, $Pe(\Gamma_m^1) \subseteq Pe(\Gamma)$. Moreover, $Pe(\Gamma_m^\infty) \subseteq Pe(\Gamma)$.*

The opposite direction of the inclusion in Proposition 2 does not hold in general. To see this, let us consider the example in Myerson [1978]. There are two players with three strategies each. There are two perfect equilibria (T, L) and (M, C); however the only equilibrium that survives maximal simultaneous deletion is (T, L).

	L	C	R
T	1,1	0,0	-9,-9
M	0,0	0,0	-7,-7
B	-9,-9	-7,-7	-7,-7

To see why, it suffices to understand that $M \succ_S B$ and that $C \succ_S R$ in Γ. Furthermore, in the game Γ^1 in which both B and R have been deleted, both $T \succ_{S^1} M$ and

$L \succ_{S^1} C$, hence only (T,L) is perfect in the fully reduced game, and it is the unique proper equilibrium of the game.

We now state the proof of Proposition 2.

Proof. Let σ be a perfect equilibrium in the game Γ_m^1. In bimatrix games, an equilibrium is perfect if and only it is undominated. An equilibrium σ is undominated if each of its components σ_i of σ is undominated. Suppose that σ is not a perfect equilibrium in $\Gamma = \Gamma_m^0$.

Either σ is not an equilibrium in Γ or σ is an equilibrium in such a game, but some of the strategies in σ are dominated in Γ. In the former case, this is a contradiction with the definition of iterated dominance, since an equilibrium σ of a reduced game is an equilibrium of the whole game. In the latter case, some of the strategies in σ are dominated in Γ so that by maximal simultaneous reduction, the strategy σ is not present in Γ_m^1, a contradiction. □

Proposition 3. *Let* Γ *be a bimatrix game satisfying* TDI^**. For any order of reduction, the set of perfect outcomes of any fully reduced game is a subset of the set of perfect outcomes of* Γ.

Proof. By Proposition 2, the set of perfect equilibria of the fully reduced game Γ_m^∞ is a subset of the set of perfect equilibria of Γ. As stated by Marx and Swinkels [1997], in any game satisfying TDI^*, any two full reductions by weak dominance are the same up to the addition or removal of redundant strategies. Moreover, the set of perfect equilibria is invariant to the addition of redundant strategies (see for instance Kohlberg and Mertens [1986]). It thus follows that the set of outcomes of any fully reduced game is a subset of the set of outcomes of the whole game. □

3.2 Finite Games

To see why Proposition 2 does not hold with more than two players, let us consider the next example (van Damme [1996] page 29).

	L	C
T	1,1,1	1,0,1
M	1,1,1	0,0,1

A

	L	C
T	1,1,0	0,0,0
M	0,1,0	1,0,0

B

In such a game, both $L \succ_S C$ and $A \succ_S B$. There is just one perfect equilibrium in Γ: (T,L,A). Nevertheless, applying maximal simultaneous reduction removes C and B from S, so that (T,L,A) and (M,L,A) are both perfect equilibria in the fully reduced game. In other words, removing weakly dominated strategies may *enlarge* the set of perfect equilibria.

Yet, the outcome is not enlarged in this example. One might wonder whether inclusion holds in terms of outcomes. The answer is again negative, as the following example shows.

This example is a modified version of the one present in Govindan and McLennan [2001], with the addition of a weakly dominated strategy X for player 3 (as long as the payoff for player 3 in each of the outcomes is strictly positive). This is an outcome game that satisfies TDI and TDI^*.

	L	R
T	a	a
M	b	b
B	a	b
D	e	f

U

	L	R
T	c	c
M	d	d
B	c	d
D	e	f

D

	L	R
T	c	c
M	d	d
B	0,0,0	0,0,0
D	e	f

X

There is a connected component of equilibria with a continuum of outcomes with support $\{T,M,B,D\}\times\{L,R\}\times\{U,D\}$. Hence, in the fully reduced game without X, this game has a continuum of perfect equilibria.

However, in the whole game, in any sequence of ε-perfect equilibria, $U_1(T,\sigma_{-1}^{\varepsilon}) > U_1(B,\sigma_{-1}^{\varepsilon})$ so that there is not a perfect equilibrium with both T and M in the support. There is not a continuum of outcomes anymore in the set of perfect equilibria. Hence, the perfect outcomes of the reduced game are a superset of the set of perfect outcomes of the whole game. Therefore, it is not even the case that $IEWDS$ restricts the set of perfect outcomes.

4 Proper Equilibria

4.1 A non-solvable game

This section presents an example showing that the proper outcomes of the whole game and of the reduced game may differ even if the TDI^* condition is satisfied. This example is a modification of the one provided by Kukushkin et al. [2008]: more precisely, two strictly dominated strategies (X and Y) have been added. The game satisfies TDI and TDI^*. There are four outcomes: a, b, c and d. Let a_i, for example, stand for the payoff for player i associated to outcome a.

	L	C	R	S
T	c	a	b	b
M	d	a	a	b
B	c	d	b	c
X	0,0	1,1	1,1	0,0
Y	1,1	0,0	0,0	1,1

Note that X and Y are strictly dominated by T, B and M as long as

$$a_1, b_1, c_1, d_1 > 1. \ (a)$$

We assume that this inequality holds. If we remove this pair of strategies, the reduced game $\Gamma^\infty = \Gamma \setminus \{X, Y\}$ has no dominated strategies. Moreover, there is a connected component $\mathcal{C}$ with a continuum of outcomes as proved by Kukushkin et al. [2008] provided that

$$d_1, b_1 < a_1, c_1 \text{ and } d_2 < b_2 < a_2, c_2, \ (b)$$

and that

$$b_2(d_1 - c_1) + b_1(c_2 - d_2) + c_1 d_2 - c_2 d_1 \neq 0. \ (c)$$

This component is defined by the following strategies:

$$\sigma_1(u_2) = \frac{1}{a_2 - b_2 + c_2 - d_2}(b_2 - d_2, c_2 - b_2, a_2 - d_2),$$

and

$$\sigma_2(u_1; t) = \left(\frac{a_1 - b_1}{a_1 - b_1 + c_1 - d_1} - \frac{(a_1 - b_1)t}{a_1 - d_1}, \frac{(c_1 - b_1)t}{a_1 - d_1} \right.$$
$$\left. \frac{c_1 - d_1}{a_1 - b_1 + c_1 - d_1} - \frac{(c_1 - d_1)t}{a_1 - d_1}, t \right).$$

We assume that (a), (b) and (c) hold, so that it is easy to check that the pair (σ_1, σ_2) defines a completely mixed strategy equilibrium in Γ^∞, provided t is positive and small enough.

We now prove that there exists a continuum of equilibria in $\mathcal{C}$ which are not proper in Γ, proving that the sets of proper equilibria of both games differ even in terms of outcomes. Note that every equilibrium in $\mathcal{C}$ is an equilibrium in Γ and is also perfect, as every undominated equilibrium is perfect in bimatrix games.

We consider the sequences $\sigma^\varepsilon = (\sigma_1^\varepsilon, \sigma_2^\varepsilon)$ of ε-proper equilibria converging towards the strategy profiles in $\mathcal{C}$.

By the definition of properness, $U_2(L, \sigma_1^\varepsilon) = U_2(S, \sigma_1^\varepsilon)$, as both are in the support of player 2's strategy. As the utility payoffs of L and S only differ when player 1 plays strategies T and M, it follows that in any ε-proper equilibrium, $\sigma_1^\varepsilon(M) = \frac{c_2 - b_2}{b_2 - d_2}\sigma_1^\varepsilon(T)$. Moreover, we must have that $U_2(C, \sigma_1^\varepsilon) = U_2(R, \sigma_1^\varepsilon)$ so that $\sigma_1^\varepsilon(B) = \frac{a_2 - b_2}{b_2 - d_2}\sigma_1^\varepsilon(T)$. Hence, it follows that $\sigma_1^\varepsilon(B) = \frac{a_2 - b_2}{c_2 - b_2}\sigma_1^\varepsilon(M)$ (*).

Finally, in any equilibrium with full support for player 2, it must be the case that $U_2(R, \sigma_1^\varepsilon) = U_2(S, \sigma_1^\varepsilon)$. This implies that

$$a_2\sigma_1^\varepsilon(M) + b_2\sigma_1^\varepsilon(B) + \sigma_1^\varepsilon(X) = b_2\sigma_1^\varepsilon(M) + c_2\sigma_1^\varepsilon(B) + \sigma_1^\varepsilon(Y).$$

Due to (*), one can check that the previous equality implies that $\sigma_1^\varepsilon(X) = \sigma_1^\varepsilon(Y)$. Hence, $U_1(X, \sigma_2^\varepsilon) = U_1(Y, \sigma_2^\varepsilon)$, as otherwise there is a contradiction with the definition of ε-properness. However, this implies that

$$\sigma_2^\varepsilon(C) + \sigma_2^\varepsilon(R) = \sigma_2^\varepsilon(L) + \sigma_2^\varepsilon(S).$$

Clearly, there exists a continuum of equilibria in $\mathcal{C}$ which do not satisfy the constraint, proving the claim.

4.2 Dominance-Solvable Games

Dominance Solvability need not imply Properness

In the following example, the unique strategy profile that survives all orders of IEWDS need not be proper. Note that the game does not satisfy TDI. Furthermore, the outcomes by dominance solvability and properness need not coincide. We focus on a bimatrix game in which each player has three strategies. Let us note that L strictly dominates C.

	L	C	R
T	2,3	1,0	0,4
M	2,2	0,0	1,-1
B	2,3	1/2,-1	1/2,4

The set of Nash equilibria equals player 1 randomizing between his three strategies, with the probability of M being higher or equal than 1/4, and player 2 playing L. Within this set, the unique pure strategy equilibrium is (M, L). Such an equilibrium is not proper, since whenever the probability of player 1 playing M becomes sufficiently close to 1, player 2 strictly prefers to play C over R. Therefore, due to the definition of ε-properness, player 1 strictly prefers to play T rather than to play M for any $\varepsilon > 0$.

Furthermore, any order of $IEWDS$ singles out the singleton (M, L). To see this, it suffices to understand that it will first remove C, then T and B (simultaneously or sequentially), and finally strategy R.

Hence, the strategy profile (M, L) satisfies three interesting features: (i) it is the unique strategy profile that survives in any order of $IEWDS$, (ii), it is not a proper equilibrium of the whole game and (iii) it does not lead to the same payoff outcome as any proper equilibrium of the whole game.

This happens because $IEWDS$ and properness choose different profiles in the Nash component. When the Nash component includes a continuum of outcomes as in this example, $IEWDS$ and properness need not induce the same outcome. One may think that this phenomenon is not surprising when we only consider dominance by pure strategies; the $IEWDS$ is a purely ordinal concept, whereas the properness depends on the expected payoffs from the deviations, hence on the cardinality of the payoffs. However, the above example does not hinge on the exact cardinality of the payoffs, in the sense that we can find a class of games with the same structure; the fact that the solution (M, L) is not proper depends on the dominance relations between the pure strategies.

Therefore, one way to ensure that both concepts lead to the same prediction is to impose a condition on the payoff structure which provides restrictions on the other players' payoffs given a deviation of a player by a mixed strategy. One such condition is TDI^*, not just TDI, since the dominance by mixed strategy matters, as in the example of Myerson [1997] (Table 5.2). The following Theorem shows that when TDI^* is combined with dominance solvability, the outcome of the Nash component is singled out, and thus the predictions by the $IEWDS$ and by the properness coincide.

A Positive Result

Before stating our main positive result, we list four properties of stable sets (see Mertens [1989] for a complete definition).

1. Stable sets always exist (*Existence*).
2. Stable sets are connected sets of normal-form perfect equilibria (*Connectedness*).
3. Stable sets of a game contain stable sets of any game obtained by deleting a pure strategy which is at its minimum probability in any normal form ε-perfect equilibrium in the neighborhood of the stable set (*Iterated Dominance and Forward Induction*).
4. Every stable set contains a proper (hence sequential) equilibrium (*Backward Induction.*).

Let us recall that the set of Nash equilibria consists of finitely many connected components (Kohlberg and Mertens [1986]).

Observation 1: Let Γ be a normal-form game that is dominance solvable while satisfying TDI^*. We let X and Y be two full reductions by weak dominance. X and Y are the same up to the addition or removal of redundant strategies (Marx and Swinkels [1997]). Moreover, since Γ is dominance solvable, there exists an order of reduction that isolates a singleton $s = (s_1, \ldots, s_n)$. Therefore, any pure strategy profile t in both X and Y satisfies $U_i(t) = U_i(s)$ for any $i \in N$.

Theorem 1. *Let Γ be a dominance-solvable game satisfying TDI^* and let s be a surviving profile. Any equilibrium with s present in its support is payoff-equivalent to s.*

Proof. Let σ be an equilibrium of Γ with $\sigma_i(s_i) > 0$, $\forall i \in N$. We have three possible cases: (**case 1**) σ is a pure-strategy equilibrium, (**case 2**) σ is a mixed-strategy equilibrium with exactly one player playing a mixed strategy, or (**case 3**) at least two players play a mixed strategy in σ.

Case 1. If σ is a pure-strategy equilibrium, then $\sigma = s$ so that $U(\sigma) = U(s)$ holds by definition.

Case 2. If σ is a mixed-strategy equilibrium in which just one player plays a mixed strategy, we let j be such a player, and hence let $\#Supp(\sigma_j) \geq 2$. It follows that $\sigma_{-j} = s_{-j}$. Therefore, $U_j(s_j, s_{-j}) = U_j(t_j, s_{-j})$ for any $s_j, t_j \in Supp(\sigma_j)$. Since TDI^* holds, it follows that $U(s_j, s_{-j}) = U(t_j, s_{-j})$, and hence $U(\sigma) = U(s)$, as wanted.

Case 3. Assume finally that σ is a mixed-strategy equilibrium in which at least two players play a mixed strategy ($\#Supp(\sigma_i) \geq 2$ for at least two players in N).

Since the game is dominance solvable and satisfies TDI^*, we know that every order of deletion o leads to a fully reduced game G_o^∞ in which all pure strategy combinations t satisfy $U(t) = U(s)$ (Observation 1). Since nice weak dominance is equivalent to weak dominance in TDI^* games, without loss of generality we can consider the order of maximal elimination e that removes at each step every nicely weakly dominated strategy. We let D_e^k denote the set of pure nicely weakly dominated strategies after k steps of elimination according to e.

3.a: If there is no nicely weakly dominated strategy in S (which is equivalent to $D_e^0 = \emptyset$), then G is a fully reduced game so that every pure-strategy profile t in S satisfies $U(t) = U(s)$. Hence $U(\sigma) = U(s)$, as wanted.

3.b: If, on the contrary, $D_e^0 \neq \emptyset$, then we let m_i in D_e^0. If m_i is in the support of σ, there are two possibilities: either $\sigma_i(m_i) = 1$ or $\sigma_i(m_i) < 1$.

If $\sigma_i(m_i) = 1$, then since $\sigma_i(s_i) > 0$ for all $i \in N$, we must have that $m_i = s_i$. Since now s_i is nicely weakly dominated, there must exist some t_i that nicely weakly dominates it in S. If $U_i(t_i, \sigma_{-i}) > U_i(s_i, \sigma_{-i})$, then s_i is not a best response, proving that σ is not an equilibrium. Hence, it must be the case that $U_i(t_i, \sigma_{-i}) = U_i(s_i, \sigma_{-i})$. However, the definition of nice weak dominance implies that if $U_i(t_i, \sigma_{-i}) = U_i(s_i, \sigma_{-i})$ then $U(t_i, \sigma_{-i}) = U(s_i, \sigma_{-i})$.

If $\sigma_i(m_i) < 1$, the equilibrium conditions imply that for every $i \in N$, $U_i(s_i, \sigma_{-i}) = U_i(m_i, \sigma_{-i})$ for any $m_i \in Supp(\sigma_i)$. However, nice weak dominance implies that if $U_i(s_i, \sigma_{-i}) = U_i(m_i, \sigma_{-i})$ then $U(s_i, \sigma_{-i}) = U(m_i, \sigma_{-i})$.

In both cases, the equilibrium payoff can be reached without the nicely weakly dominated strategy in the support.

3.c: Given that the payoff of σ does not depend on any nicely weakly dominated strategy in D_e^0, it must depend on the strategies in $S \setminus D_e^0$.

We let S^1 be the restriction $S \setminus D_e^0$. Note that the game $G^1 = (S^1, u)$ is the one obtained after one step of removing all nicely weakly dominated strategies.

3.d: If there are no nicely weakly dominated strategies in this restriction (i.e.$D_e^1 = \emptyset$), then the game is fully reduced so that every pure-strategy combination t satisfies $U(t) = U(s)$, and hence $U(\sigma) = U(s)$, as wanted.

3.e: If, on the contrary, $D_e^1 \neq \emptyset$, the equilibrium payoff can be attained without the nicely weakly dominated strategies. Since the game is dominance solvable, iterating this procedure until no nicely weakly dominated strategy is left leads to a game in which any pure-strategy combination has the same payoff as s, proving that $U(\sigma) = U(s)$, as wanted.

□

Theorem 2. *Let Γ be a dominance-solvable game that satisfies TDI^*. Then:*

(i) *Under any order of iterative elimination of weakly dominated strategies, the outcome is the unique stable one.*

(ii) *For any equilibrium of the fully reduced game, there is a proper equilibrium of Γ which induces the same outcome.*

Proof. Since the game satisfies TDI^*, all fully reduced games lead to the payoff associated with s, the surviving singleton. Moreover, the inclusion property of Mertens sets ensures that s is stable. Hence, there must exist some proper equilibrium in G with payoff identical to s (Backwards Induction property of stable sets). In addition, since some order isolates s, then there is at most one stable set. Finally, all equilibria in the component of s lead to the same payoff (Theorem 1). Hence, the outcome of s is the unique stable one. □

Why do we need TDI^* rather than TDI?

The main logic behind Theorem 1 is that all orders of deletion are equivalent under TDI^*. More specifically, nice weak dominance and weak dominance coincide whenever the game satisfies TDI^*. The proof of the theorem relies on the fact that (iteratively) applying nice weak dominance does not enlarge the set of Nash payoffs. Does the same result hold if we only apply TDI?

Suppose that a singleton is selected by some order of $IEWDS$. The outcome of this singleton must coincide with that of a proper equilibrium of the whole game if this precise order satisfies nice weak dominance (i.e. all removed strategies are nicely weakly dominated). Yet, the set of proper outcomes might be enlarged by applying $IEWDS$ in a game satisfying TDI but not TDI^*, as shown by the next example, related to the one provided by Marx and Swinkels [1997] (p.233).

	L	C	R
T	2,1	4,3	0,2
M	0,3	3,1	4,2
B	1,4	1,4	1,4
D	1,4	0,3	0,2

This game satisfies TDI but not TDI^*. Indeed, the strategy R is very weakly dominated by $1/2L + 1/2C$ in $S \setminus \{D, B\}$, but is neither weakly dominated nor redundant on $S \setminus \{D, B\}$. Moreover, this game is dominance solvable. After eliminating R, the strategies M, B and D are strictly dominated by T. Then, eliminating L leads to (T, C) as the surviving profile.

On the contrary, if we only eliminate D and B from S, then we are left with the fully reduced game $\{T, M\} \times \{L, C, R\}$. In this game, there is a set of completely mixed strategy equilibria (hence proper) of the following type:

$$(1/2T + 1/2M, (pL + qC + (1-p-q)R)), \text{ as long as } 6p + 5q = 4.$$

However, some equilibria of this set are not proper equilibria of the whole game. Indeed, note that as far as R is in the support of an equilibrium (take for instance the equilibrium in which $p = \varepsilon$ and $q = (4-6\varepsilon)/5$), this equilibrium cannot be proper in the whole game, since R is weakly dominated by $1/2L + 1/2R$ in S. Therefore, the set of proper equilibria might be enlarged by $IEWDS$ in a dominance-solvable game that satisfies TDI but fails to satisfy TDI^*.

5 Conclusion

In this paper we explore the conditions under which simplification of the game by $IEWDS$ can be applied to analyze strategic stability of the equilibria.

It turns out that, surprisingly, neither the TDI^* condition of Marx and Swinkels [1997] nor dominance solvability alone is sufficient to guarantee that the set of proper equilibria of the reduced game is included in the set of proper equilibria of the whole game (proper inclusion). Our examples show that the negative results are obtained even in terms of the equilibrium *outcome*.

We elaborated on finding examples in which the TDI^* condition alone is not sufficient; indeed $IEWDS$ may *enlarge* the set of proper outcomes. Dominance solvability alone is not sufficient either: we give an example in which the outcome singled out by dominance solvability does not coincide with any proper outcome of the whole game.

If the game satisfies *both* TDI^* and dominance solvability, we show that proper inclusion holds. Moreover, the uniqueness of the stable outcome is guaranteed.

There is a large class of games for which our sufficient conditions are satisfied (see a recent work by Milgrom and Segal [2014] on deferred-acceptance auctions). For example, in many strategic interactions in political competition, such as voting, players' payoffs depend solely on the outcome, which is determined by the social choice, such as the winner of the election. The TDI^* condition is relevant in many situations (see Marx and Swinkels [1997]). Even in games in which the DS condition is not satisfied, if the outcome is isolated, proper inclusion is guaranteed. We can safely apply $IEWDS$ to simplify the game and analyze the strategic stability of the whole game by focusing on the reduced game.

This paper provides a set of sufficient conditions under which we can take advantage of both the simplicity of $IEWDS$ and the robustness of strategic stability.

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