

# Bidirectional A\* Search on Time-Dependent Road Networks

Giacomo Nannicini

LIX, École Polytechnique, F-91128 Palaiseau, France, and  
Mediamobile, 27 boulevard Hippolyte Marquès, 94200 Ivry sur Seine, France

Daniel Delling and Dominik Schultes

Universität Karlsruhe (TH), 76128 Karlsruhe, Germany

Leo Liberti

LIX, École Polytechnique, F-91128 Palaiseau, France

**The computation of point-to-point shortest paths on time-dependent road networks has a large practical interest, but very few works propose efficient algorithms for this problem. We propose a novel approach, which tackles one of the main complications of route planning in time-dependent graphs, which is the difficulty of using bidirectional search: because the exact arrival time at the destination is unknown, we start a backward search from the destination node using lower bounds on arc costs to restrict the set of nodes that have to be explored by the forward search. Our algorithm is based on A\* with landmarks (ALT); extensive computational results show that it is very effective in practice if we are willing to accept a small approximation factor, resulting in a speed-up of more than one order of magnitude with respect to Dijkstra's algorithm while finding only slightly suboptimal solutions. The main idea presented here can also be generalized to other types of search algorithms. © 2011 Wiley Periodicals, Inc. NETWORKS, Vol. 59(2), 240–251 2012**

**Keywords:** shortest paths; time-dependent networks; road networks; goal directed search

## 1. INTRODUCTION

Route planning in road networks is a practical application that, in recent years, has attracted a lot of attention to the computation of shortest paths on large graphs. In particular, since in several countries there are now road segments covered with traffic sensors, it is possible to generate speed profiles

based on historical data. It thus becomes feasible to model the dependence of travelling speed on the time of the day; consequently, situations like rush hour traffic peaks can be taken into account during the calculation, giving much more meaningful results with respect to the static case (where arc costs are always fixed) from the user's point of view. In a typical application scenario, for example, a server machine, which provides a route planning web server, one would like to answer several shortest path queries in less than 1 s of CPU time, on graphs with several million nodes. This means that we are interested in an algorithm which is able to quickly find good solutions to the time-dependent shortest path problem, which we define as follows.

Given a set of time instants  $T = \mathbb{R}_+$ , consider a directed graph  $G = (V, A)$  equipped with a time-dependent arc weight function  $c : A \times T \rightarrow \mathbb{R}_+$ , such that for each pair of time instants  $\tau, \tau' \in T$  with  $\tau < \tau'$  the property  $\forall (u, v) \in A \ c(u, v, \tau) + \tau \leq c(u, v, \tau') + \tau'$  holds. This condition on  $c$  is known as the FIFO property. The FIFO property is also called the nonovertaking property: it basically says that if  $T_1$  leaves  $u$  at time  $\tau$  and  $T_2$  leaves at time  $\tau' > \tau$ ,  $T_2$  cannot arrive at  $v$  before  $T_1$  using the arc  $(u, v)$ . Given a path  $p = (s = v_1, \dots, v_k = t)$  in  $G$ , its time-dependent cost  $\gamma_{\tau_0}(p)$  is defined recursively as:

$$\gamma_{\tau_0}(v_1, v_2) = c(v_1, v_2, \tau_0) \quad (1)$$

$$\gamma_{\tau_0}(v_1, \dots, v_i) = \gamma_{\tau_0}(v_1, \dots, v_{i-1}) + c(v_{i-1}, v_i, \tau_0 + \gamma_{\tau_0}(v_1, \dots, v_{i-1})) \quad (2)$$

for all  $2 \leq i \leq k$ . We can now formally introduce the problem discussed in this article.

Received May 2008; accepted February 2011

Correspondence to: G. Nannicini; e-mail: giacomon@lix.polytechnique.fr

DOI 10.1002/net.20438

Published online 27 May 2011 in Wiley Online Library (wileyonlinelibrary.com).

© 2011 Wiley Periodicals, Inc.

TIME-DEPENDENT SHORTEST PATH PROBLEM (TDSPP): Given  $G = (V, A)$ ,  $T$  and  $c$  as defined above, a source node  $s \in V$ , a destination node  $t \in V$ , and a starting time  $\tau_0 \in T$ , find a path  $p = (s = v_1, \dots, v_k = t)$  in  $G$  such that  $\gamma_{\tau_0}(p)$  is minimum.

For the TDSPP, the FIFO assumption is usually necessary to maintain polynomial complexity: the SPP in time-dependent FIFO networks is polynomially solvable [26], whereas it is NP-hard in non-FIFO networks [31].

We assume that a function  $\lambda : A \rightarrow \mathbb{R}_+$  with the following property:

$$\forall (u, v) \in A, \tau \in T \quad \lambda(u, v) \leq c(u, v, \tau),$$

is known. In other words,  $\lambda(u, v)$  is a lower bound on the travelling time of arc  $(u, v)$  for all time instants in  $T$ . In practice, this can easily be computed given an arc length and the maximum allowed speed on that arc. We naturally extend  $\lambda$  to be defined on paths, that is,  $\lambda(p) = \sum_{(v_i, v_j) \in p} \lambda(v_i, v_j)$ . We call  $G_\lambda$  the graph  $G$  weighted by the lower bounding function  $\lambda$ .

In this article, we propose a novel algorithm for the TDSPP on FIFO networks based on a bidirectional  $A^*$  algorithm. As the arrival time is not known in advance (so  $c$  cannot be evaluated on the arcs adjacent to the destination node), our backward search occurs on  $G_\lambda$  and is therefore a time-independent search. This is used for bounding the set of nodes that will be explored by the forward search. An extended abstract of this work appeared in [29], which represented the first attempt to tackle the TDSPP in a bidirectional fashion. Since then, our idea has been used for several shortest paths algorithms on time-dependent networks (see e.g., [3, 11]).

### 1.1. Related Work

Many ideas have been proposed for the computation of point-to-point shortest paths on static graphs (see [12] for a review), and there are algorithms capable of finding the solution in a matter of a few microseconds [2]; adaptations of those ideas for dynamic scenarios, that is, where arc costs are updated at regular intervals, have been tested as well [13, 28, 33, 35] (see [30] for a survey).

Much less work has been undertaken on the time-dependent variant of the shortest paths problem. The TDSPP was first addressed in [8]: a recursive formula is given to establish the minimum time to travel to a given target starting from a given source at a certain time  $\tau$ . In [16], Dijkstra's algorithm [15] is extended to the dynamic case, but the FIFO property, which is necessary to prove that Dijkstra's algorithm terminates with a correct shortest paths tree on time-dependent networks, is not mentioned. As Dijkstra's algorithm plays an important role in this article, we review it in Section 2. Given source and destination nodes  $s$  and  $t$ , the problem of maximizing the departure time from node  $s$  with a given arrival time at node  $t$  is equivalent to the TDSPP (see [9]). A survey on the TDSPP is given in [14].

Goal-directed search, also called  $A^*$  [24], has been adapted to work on all the previously described scenarios; an efficient version for the static case has been presented in [20], and then developed and improved in [21]. Those ideas have been used in [13] on dynamic graphs as well, whereas the time-dependent case on graphs with the FIFO property has been addressed in [7] and [13].

After the publication of the extended abstract of this work [29], several speed-up techniques have been adapted to the time-dependent scenario. The SHARC-algorithm [4] allows fast unidirectional shortest path calculations in large scale networks. Because of its unidirectional nature, it can easily be used in a time-dependent scenario [10]. Moreover, contraction hierarchies [19] have been adapted as well [3], but the memory consumption of this approach is very large. Finally, in [11] the approach introduced in this work is enhanced with an exact bi-level search method (i.e., most of the search is carried out on a smaller network that plays the same role as motorways in real-life road networks). The resulting algorithm, TDCALT (time-dependent core-ALT) [11], is one of the fastest known techniques for route planning in time-dependent road networks.

### 1.2. Overview

The rest of this article is organized as follows. We review Dijkstra's algorithm in Section 2. In Section 3, we describe  $A^*$  search and the ALT algorithm, which are needed for our method. In Section 4, we provide the foundations of our idea in a simple way by using Dijkstra's algorithm. In Section 5, we adapt those ideas to the ALT algorithm, giving a specific implementation. We formally prove our method's correctness in Section 6 for both exact and approximated shortest path computations. In Section 7, we propose some modifications that improve the performance of our algorithm and prove their correctness. Computational experiments in Section 8 show the feasibility of our approach.

## 2. DIJKSTRA'S ALGORITHM

Dijkstra's algorithm [15] solves the single source SPP in static directed graphs with non-negative weights in polynomial time. The algorithm can easily be generalized to the time-dependent case [16]. Dijkstra's algorithm is a labeling method.

The labeling method for the SPP [17] finds shortest paths from the source to all vertices in the graph; on a static graph with arc weights  $w(u, v) \forall (u, v) \in A$ , the method works as follows: for every vertex  $v$ , it maintains its distance label  $\ell[v]$ , parent node  $p[v]$ , and status  $S[v]$  which may be one of the following: *unreached*, *explored*, and *settled*. Initially  $\ell[v] = \infty$ ,  $p[v] = NIL$ , and  $S[v] = unreached$  for every vertex  $v$ . The method starts by setting  $\ell[s] = 0$  and  $S[s] = explored$ ; while there are labeled (i.e., *explored*) vertices, the method picks an *explored* vertex  $u$ , relaxes all outgoing arcs of  $u$ , and sets  $S[u] = settled$ . To relax an arc  $(u, v)$ , one checks if  $\ell[v] > \ell[u] + w(u, v)$  and, if true, sets

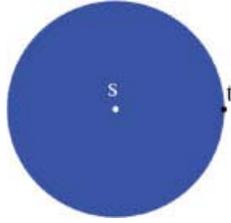


FIG. 1. Schematic representation of Dijkstra's algorithm search space. [Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

$\ell[v] = \ell[u] + w(u, v)$ ,  $p[v] = u$ , and  $S[v] = \text{explored}$ . At any iteration of the algorithm, the set of nodes with status *explored* or *settled* is called the *search scope*. If the graph does not contain cycles with negative cost, the labeling method terminates with correct shortest path distances and a shortest path tree. The algorithm can be extended to the time-dependent case on FIFO networks by a simple modification of the arc relaxation procedure: if  $\tau_0$  is the departure time from the source node, we check if  $\ell[v] > \ell[u] + c(u, v, \tau_0 + \ell[u])$  and, if true, set  $\ell[v] = \ell[u] + c(u, v, \tau_0 + \ell[u])$ ,  $p[v] = u$ , and  $S[v] = \text{explored}$ . The efficiency of the label-setting method depends on the rule to choose a vertex to scan next. We say that  $\ell[v]$  is exact if it is equal to the distance from  $s$  to  $v$ ; it is easy to see that if one always selects a vertex  $u$  such that  $\ell[u]$  is exact at the selection time, then each vertex is scanned at most once. In this case, we only need to relax arcs  $(u, v)$  where  $v$  is not *settled*, and the algorithm is called *label-setting*. Dijkstra [15] observed that if the cost function  $c$  is non-negative and  $v$  is an explored vertex with the smallest distance label, then  $\ell[v]$  is exact; so, we refer to the labeling method with the minimum label selection rule as Dijkstra's algorithm. If  $w(u, v)$  is non-negative  $\forall (u, v) \in A$ , then Dijkstra's algorithm scans vertices in non-decreasing order of distance from  $s$  and scans each vertex at most once; for the point-to-point SPP, we can terminate the labeling method as soon as the target node is *settled*. The algorithm requires  $O(|A| + |V| \log |V|)$  amortized time if the queue is implemented as a Fibonacci heap [18]; with a binary heap, the running time is  $O((|E| + |V|) \log |V|)$ . Note that on road networks, we typically have  $|E| = O(|V|)$ ; therefore, the running time of Dijkstra's algorithm with binary heaps is  $O(|V| \log |V|)$ .

One basic variant of Dijkstra's algorithm for the point-to-point SPP is bidirectional search; instead of building only one shortest path tree rooted at the source node  $s$ , we also build a shortest path tree rooted at the target node  $t$  on the reverse graph (i.e., the graph  $\bar{G} = (V, \bar{A})$  where  $(u, v) \in \bar{A} \Leftrightarrow (v, u) \in A$ ). As soon as one node  $v$  becomes *settled* in both searches, we are guaranteed that the shortest  $s \rightarrow t$  path has been found: there is a node  $u$  *settled* by the forward search and a node  $w$  *settled* by the backward search such that the concatenation of the shortest  $s \rightarrow u$  path and the shortest  $w \rightarrow t$  path plus the arc  $(u, w) \in A$  forms the shortest  $s \rightarrow t$  path. As we can think of Dijkstra's algorithm as exploring nodes in circles centered at  $s$  with increasing radius until  $t$  is

reached (see Fig. 1), the bidirectional variant is faster because it explores nodes in two circles centered at both  $s$  and  $t$ , until the two circles meet (see Fig. 2); the area within the two circles, which represents the number of explored nodes, will then be smaller than in the unidirectional case, up to a factor of two.

Dijkstra's algorithm applied to time-dependent FIFO networks has been optimized in various ways [5, 6]. We note here that in the time-dependent scenario bidirectional search cannot be applied, because the arrival time at destination node is unknown. We also remark that all speedup techniques based on finding shortest paths in Euclidean graphs [23, 34] cannot be applied either, because the typical arc cost function, the arc travelling time at a certain time of the day, does not yield a Euclidean graph.

### 3. A\* WITH LANDMARKS

A\* is an algorithm for goal-directed search that is similar to Dijkstra's algorithm; the main difference lies in the fact that A\* adds a potential function to the priority key of each node in the queue. The potential function on a node  $v$  is an estimate of the distance to reach the target from  $v$ ; A\* then follows the same procedure as Dijkstra's algorithm, but the use of this potential function, summed to the priority key of each node, has the effect of prioritizing nodes that are likely to be closer to the target node  $t$ . If the potential function  $\pi$  is such that  $\pi(v) \leq d(v, t) \forall v \in V$ , where  $d(v, t)$  is the distance from  $v$  to  $t$ , then A\* always finds shortest paths. If  $\pi(v)$  is a good approximation from below of the distance to target, A\* efficiently drives the search toward the destination node, and it explores considerably fewer nodes than Dijkstra's algorithm; if  $\pi(v) = 0 \forall v \in V$ , A\* behaves exactly like Dijkstra's algorithm, that is, it explores the same nodes. A\* is equivalent to Dijkstra's algorithm on a graph with reduced costs  $w_\pi(u, v) = w(u, v) - \pi(u) + \pi(v)$  (see e.g. [1, 25]); as the length of each path between  $s$  and  $t$  changes by the same amount  $\pi(t) - \pi(s)$ , the shortest path is invariant. Note that since Dijkstra's algorithm requires arc costs to be non-negative, the potential function should be consistent, that is,  $\pi(u) \leq w(u, v) + \pi(v) \forall (u, v) \in A$ .

One way to compute the potential function, instead of using Euclidean distances, is to use the concept of landmarks. Landmarks were first proposed in [20]; they are a preprocessing technique, which is based on the triangular inequality. The basic principle is as follows: suppose we have selected a set  $L \subset V$  of landmarks, and we have precomputed distances

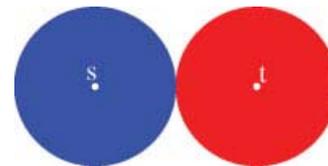


FIG. 2. Schematic representation of bidirectional Dijkstra's algorithm search space. [Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

$d(v, \ell), d(\ell, v) \forall v \in V, \ell \in L$ ; the following triangle inequalities hold:  $d(u, t) + d(t, \ell) \geq d(u, \ell)$  and  $d(\ell, u) + d(u, t) \geq d(\ell, t)$ . Therefore,  $\pi_t(u) = \max_{\ell \in L} \{d(u, \ell) - d(t, \ell), d(\ell, t) - d(\ell, u)\}$  is a lower bound for the distance  $d(u, t)$ , and it can be used as a potential function which preserves optimal paths. On static (i.e., non-time-dependent) graphs, landmarks can be used to implement bidirectional search, using some care in modifying the potential function so that it is consistent for both forward and backward search [21]. This translates to ensuring that  $w_{\pi_f}(u, v)$  in  $G$  is equal to  $w_{\pi_b}(v, u)$  in the reverse graph  $\bar{G}$ , where  $\pi_f$  and  $\pi_b$  are the potential functions for the forward and the backward search, respectively. Bidirectional  $A^*$  with the potential function described above is called ALT. It is straightforward to note that, if arc costs can only increase with respect to their original value, that is, the value used in the precomputation of landmark distances, then the potential function associated to landmarks is still a valid lower bound, even on a time-dependent graph. In [13], this idea is applied to a real road network to analyze algorithmic performances but with a unidirectional search. On road networks, the initial arc cost, which should be a lower bound on the time-dependent cost on that arc, can be easily computed by dividing the arc's length by the maximum allowed speed on that arc's road category.

The choice of landmarks has a great impact on the size of the search space, as it severely affects the quality of the potential function. Several selection strategies exist, although none of them is optimal with respect to random queries, in the sense that none is guaranteed to yield the smallest search space for random source-destination pairs. The best known heuristics are Avoid and MaxCover [20, 22].

#### 4. BIDIRECTIONAL SEARCH ON TIME-DEPENDENT GRAPHS

We assume that we are given a graph  $G = (V, A)$ , source and destination vertices  $s, t \in V$ , and a departure time  $\tau_0 \in T$ . In the rest of this article, we denote by  $d(u, v, \tau)$  the length of the shortest path from  $u$  to  $v$  with departure time  $\tau$ , and by  $d_\lambda(u, v)$  the length of the shortest path from  $u$  to  $v$  on the graph  $G_\lambda$ . The approach that we propose for bidirectional search on time-dependent graphs is based on a modification of Dijkstra's algorithm.

For any  $u, v, \in V, \tau \in T$ , let  $l(u, v, \tau) \leq d(u, v, \tau)$  be any lower bounding function for the distance between  $u$  and  $v$  with departure time  $\tau$ . Assume that we have an upper bound  $\mu$  on the cost of the optimal solution to TDSPP; for example,  $\mu$  is the time-dependent cost of any  $s \rightarrow t$  path with departure time  $\tau_0$ . We run Dijkstra's algorithm with the following pruning criterion: eliminate any unsettled node  $v$  for which

$$\max\{d(s, u, \tau_0) : u \text{ is settled}\} + l(v, t, \max\{d(s, u, \tau_0) : u \text{ is settled}\}) > \mu. \quad (3)$$

In the following, the term "pruning" stands for "do not insert in the priority queue." Proposition 4.1 establishes correctness of our approach.

**Proposition 4.1.** *Nodes satisfying (3) are not necessary to compute the shortest path from  $s$  to  $t$  with departure time  $\tau_0$  using Dijkstra's algorithm.*

**Proof.** Suppose, at some iteration of Dijkstra's algorithm, that the pruning criterion (3) eliminates some nodes that are on the shortest path  $p^*$  from  $s$  to  $t$  with departure time  $\tau_0$ . Let  $v$  be the first of these nodes. Then  $v$  is such that  $d(s, v, \tau_0) \geq \max\{d(s, u, \tau_0) : u \text{ is settled}\}$ . Hence,  $\gamma_{\tau_0}(p^*) \leq \mu < \max\{d(s, u, \tau_0) : u \text{ is settled}\} + l(v, t, \max\{d(s, u, \tau_0) : u \text{ is settled}\}) \leq d(s, v, \tau_0) + l(v, t, d(s, v, \tau_0)) \leq \gamma_{\tau_0}(p^*)$ , which is a contradiction. For the last inequality in the chain, we need the FIFO property. ■

This far, the algorithm looks unidirectional, and we did not specify how the lower bounds  $l(u, v, \tau)$  can be obtained. We use bidirectional search to this end. Our proposal is as follows: run a backward search from  $t$  on  $G_\lambda$ . For each node  $v$  settled by the backward search,  $d_\lambda(v, t) \leq d(v, t, \tau) \forall \tau \in T$ ; hence, we can use  $l(v, t, \tau) = d_\lambda(v, t)$ . Therefore, the backward search's purpose is to provide bounds for the pruning criterion of the forward search, which is the only search that uses time-dependent costs.

We still need to specify several missing details: how do we obtain  $\mu$ ? How do we choose between performing forward or backward search iterations, and when should the backward search be stopped? Furthermore, we can see intuitively that a straightforward implementation of this algorithm is not likely to be useful in practice, because we would need to perform an extensive backward search (i.e., explore a large portion of the graph around the target, even in the "wrong" direction) before we are able to effectively prune the forward search. In the next section, we tackle all this issues by using the  $A^*$  algorithm instead of Dijkstra's algorithm.

#### 5. BIDIRECTIONAL SEARCH WITH $A^*$

In Section 4, we have described a general framework for bidirectional search on time-dependent graphs. In this section, we fill in the missing details, giving a description that can be implemented in practice and use the  $A^*$  algorithm instead of Dijkstra's algorithm; recall that  $A^*$  is a generalization of Dijkstra's algorithm, in that Dijkstra's algorithm is equivalent to  $A^*$  with a zero potential function.

The algorithm for computing the shortest time-dependent cost path  $p^*$  works in three phases.

1. A bidirectional  $A^*$  search occurs on  $G$ , where the forward search is run on the graph weighted by  $c$  with the path cost defined by (1) and (2), and the backward search is run on  $G_\lambda$ . All nodes settled by the backward search are included in a set  $M$ . Phase 1 terminates as soon as the two search scopes meet.
2. Suppose that  $v \in V$  is the first vertex to be explored by both forward and backward search; let  $\mu = \gamma_{\tau_0}(p_v)$ , where  $p_v$  is the path from  $s$  to  $t$  passing through  $v$ . In the second phase, both searches are allowed to proceed until the backward search queue only contains nodes whose

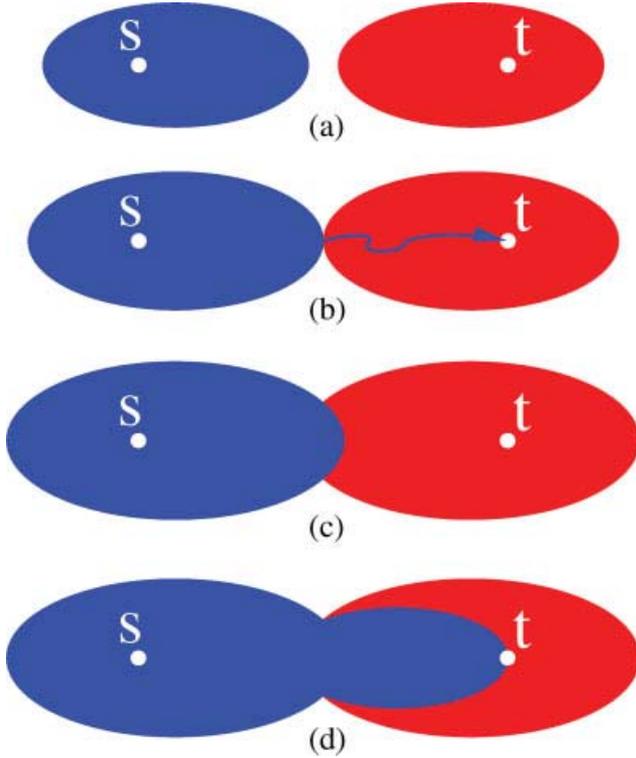


FIG. 3. Schematic representation of the TDALT algorithm's search space. [Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

associated key exceeds  $\mu$ . In other words, let  $\beta$  be the key of the minimum element of the backward search queue. Phase 2 terminates as soon as  $\beta > \mu$ . Again, all nodes settled by the backward search are included in  $M$ .

3. Only the forward search continues, with the additional constraint that only nodes in  $M$  can be explored. The forward search terminates when  $t$  is settled.

The pseudocode for this algorithm is given in Algorithm 1. Note that we use the symbol  $\leftrightarrow$  to indicate either the forward search ( $\leftrightarrow = \rightarrow$ ) or the backward search ( $\leftrightarrow = \leftarrow$ ). We denote by  $\vec{A}$  the set of arcs for the forward search, that is,  $\vec{A} = A$ , and by  $\overleftarrow{A}$  the set of arcs for the backward search, i.e.,  $\overleftarrow{A} = \{(u, v) | (v, u) \in A\}$ . A typical choice is to alternate between the forward and the backward search at each iteration of the algorithm during the first two phases. Algorithm 1 works with any choice of feasible potential function; but since we use landmark-based potentials (see Section 3), we call this algorithm time-dependent ALT (TDALT). A schematic representation of the different phases is given in Figure 3.

It is easy to see how TDALT is an instantiation that employs  $A^*$  of the idea discussed in Section 4. In Phase 1, we use the backward search to find an  $s \rightarrow t$  path in order to compute an upper bound  $\mu$  to the optimal cost of the solution. In Phase 2, we alternate between the two searches in order to fulfill the pruning criterion (3). Phase 2 ends once (3) is satisfied for all nodes that are not in  $M$ . At this point, we only need to explore nodes in  $M$ .

---

**Algorithm 1** TIME-DEPENDENT ALT: Compute the shortest time-dependent path from  $s$  to  $t$  with departure time  $\tau_0$

---

```

1:  $\vec{Q}.$ insert( $s, 0$ );  $\overleftarrow{Q}.$ insert( $t, 0$ );  $M := \emptyset$ ;  $\mu := +\infty$ ;
    $done := false$ ;  $phase := 1$ .
2: while  $\neg done$  do
3:   if ( $phase = 1$ )  $\vee$  ( $phase = 2$ ) then
4:      $\leftrightarrow \in \{\rightarrow, \leftarrow\}$ 
5:   else
6:      $\leftrightarrow := \rightarrow$ 
7:      $u := \vec{Q}.$ extractMin()
8:     if ( $u = t$ )  $\wedge$  ( $\leftrightarrow = \rightarrow$ ) then
9:        $done := true$ 
10:    continue
11:    if ( $phase = 1$ )  $\wedge$  ( $u.dist^{\rightarrow} + u.dist^{\leftarrow} < \infty$ ) then
12:       $\mu := u.dist^{\rightarrow} + u.dist^{\leftarrow}$ 
13:       $phase := 2$ 
14:    if ( $phase = 2$ )  $\wedge$  ( $\leftrightarrow = \leftarrow$ )  $\wedge$  ( $\mu < u.key^{\leftarrow}$ ) then
15:       $phase := 3$ 
16:    continue
17:    for all arcs  $(u, v) \in \overleftarrow{A}$  do
18:      if  $\leftrightarrow = \leftarrow$  then
19:         $M.$ insert( $u$ )
20:      else if ( $phase = 3$ )  $\wedge$  ( $v \notin M$ ) then
21:        continue;
22:      if ( $v \in \overleftarrow{Q}$ ) then
23:        if  $u.dist^{\leftrightarrow} + c(u, v, u.dist^{\leftrightarrow}) < v.dist^{\leftrightarrow}$  then
24:           $\overleftarrow{Q}.$ decreaseKey( $v, u.dist^{\leftrightarrow} + c(u, v, u.dist^{\leftrightarrow}) + \overleftarrow{\pi}(v)$ )
25:        else
26:           $\overleftarrow{Q}.$ insert( $v, u.dist^{\leftrightarrow} + c(u, v, u.dist^{\leftrightarrow}) + \overleftarrow{\pi}(v)$ )
27: return  $t.dist^{\rightarrow}$ 

```

---

## 6. CORRECTNESS

We prove correctness of Algorithm 1. Intuitively, this follows from Proposition 4.1 by applying the necessary modifications to deal with  $A^*$  instead of Dijkstra's algorithm. However, we give a slightly different proof, which makes explicit reference to  $\beta$ , because we will use the same technique to prove other claims in the remainder of the paper.

**Theorem 6.1.** *Algorithm 1 computes the shortest time-dependent path from  $s$  to  $t$  for a given departure time  $\tau_0$ .*

**Proof.** The forward search of Algorithm 1 is exactly the same as the unidirectional version of the  $A^*$  algorithm during the first two phases, and thus, it is correct; we just have to prove that the restriction applied during Phase 3 does not interfere with the correctness of the  $A^*$  algorithm, that is, that we do not prune nodes on the shortest path.

Let  $\mu$  be an upper bound on the cost of the shortest path; in particular, this can be the cost  $\gamma_{\tau_0}(p, v)$  of the  $s \rightarrow t$  path passing through the first meeting point  $v$  of the forward and backward search. Let  $\beta$  be the smallest key of the backward search priority queue at the end of Phase 2. Let  $p^*$  be the

shortest path from  $s$  to  $t$  with departure time  $\tau_0$ , and suppose there are some nodes on  $p^*$ , which are not settled by the forward search. Let  $u$  be the first of these nodes; this implies that  $u \notin M$ , that is,  $u$  has not been settled by the backward search during the first two phases of Algorithm 1. Hence, we have that  $\beta \leq \pi_b(u) + d_\lambda(u, t)$ ; then we have the chain  $\gamma_{\tau_0}(p^*) \leq \mu < \beta \leq \pi_b(u) + d_\lambda(u, t) \leq d_\lambda(s, u) + d_\lambda(u, t) \leq d(s, u, \tau_0) + d(u, t, d(s, u, \tau_0)) = \gamma_{\tau_0}(p^*)$ , which is a contradiction. ■

**Theorem 6.2.** *Let  $p^*$  be the shortest path from  $s$  to  $t$ . If the condition to switch to Phase 3 is  $\mu < K\beta$  for a fixed parameter  $K$ , then Algorithm 1 computes a path  $p$  from  $s$  to  $t$  such that  $\gamma_{\tau_0}(p) \leq K\gamma_{\tau_0}(p^*)$  for a given departure time  $\tau_0$ .*

**Proof.** Suppose that  $\gamma_{\tau_0}(p) > K\gamma_{\tau_0}(p^*)$ . Let  $u$  be the first node on  $p^*$  that is not explored by the forward search; by Phase 3, this implies that  $u \notin M$ , that is,  $u$  has not been settled by the backward search during the first two phases of Algorithm 1. Hence, we have that  $\beta \leq \pi_b(u) + d_\lambda(u, t)$ ; then we have the chain  $\gamma_{\tau_0}(p) \leq \mu < K\beta \leq K(\pi_b(u) + d_\lambda(u, t)) \leq K(d_\lambda(s, u) + d_\lambda(u, t)) \leq K(d(s, u, \tau_0) + d(u, t, d(s, u, \tau_0))) = K(\gamma_{\tau_0}(p^*)) < \gamma_{\tau_0}(p)$ , which is a contradiction. ■

## 7. IMPROVEMENTS

Performance of the basic version of the algorithm can be improved with the results that we describe in this section.

**Theorem 7.1.** *Let  $p^*$  be the shortest path from  $s$  to  $t$  with departure time  $\tau_0$ . If all nodes  $u$  on  $p^*$  settled by the backward search are settled with a key smaller or equal to  $d(s, u, \tau_0) + d(u, t, d(s, u, \tau_0))$ , then Algorithm 1 is correct.*

**Proof.** Let  $Q$  be the backward search queue; let  $\text{key}(u)$  be the key for the backward search of node  $u$ ; let  $\beta = \text{key}(v)$  be the smallest key in the backward search queue when Phase 2 ends, which is attained at a node  $v$ ; and let  $\mu$  the best upper bound on the cost of the solution currently known. To prove correctness, using the same arguments as in the proof of Theorem 6.1, we must make sure that when the backward search stops at the end of Phase 2 then all nodes on the shortest path from  $s$  to  $t$  that have not been explored by the forward search have been added to  $M$ . The backward search stops when  $\mu < \beta$ .

In an  $A^*$  search on FIFO networks, the keys of settled nodes are nondecreasing. So every node  $u$  on  $p^*$  that has not been settled by the backward search at the end of Phase 2 would be settled with a key  $\text{key}(u) \geq \text{key}(v)$ , which yields  $d(s, u, \tau_0) + d(u, t, d(s, u, \tau_0)) \geq \text{key}(v) = \beta > \mu \forall u \in Q$ . Thus,  $u$  cannot be on the shortest path from  $s$  to  $t$ , and Algorithm 1 is correct. ■

This allows the use of larger lower bounds during the backward search: the backward  $A^*$  search does not have to compute shortest paths on the graph  $G_\lambda$ , but it should in any case guarantee that when a node  $u$  is settled, its key is

an underestimation of the time-dependent cost of the time-dependent shortest path between  $s$  and  $t$  passing through  $u$ . This is similar to what is required by the modified Dijkstra's algorithm of Section 4.

The next proposition is of fundamental practical importance; it states that the backward search can be pruned at nodes already settled by the forward search, because all nodes that are on the shortest path and that have not been settled by the forward search can be reached by the backward search through another path.

**Proposition 7.2.** *During Phase 2 the backward search does not need to explore nodes that have already been settled by the forward search.*

**Proof.** Suppose that the forward search has settled all nodes on the shortest  $s \rightarrow t$  path  $p^*$  up to node  $u$ . Then clearly all remaining nodes on  $p^*$  are reachable from  $t$  in the backward search with a path that does not use any node already settled by the forward search (i.e., the subpath of  $p^*$  from  $u$  to  $t$ ). Therefore, the backward search can be pruned at all nodes settled by the forward search. ■

By Theorem 7.1, we can take advantage of the fact that the backward search is used only to bound the set of nodes explored by the forward search. This means that we can tighten the bounds used by the backward search: the potential function for the backward search does not have to be feasible. To derive some valid lower bounds we need the following proposition.

**Proposition 7.3.** *At a given iteration, let  $v$  be any node settled by the forward search. Then, for each node  $w$  which has not been settled by the forward search,  $d(s, v, \tau_0) + \pi_f(v) - \pi_f(w) \leq d(s, w, \tau_0)$ .*

**Proof.** As the forward search uses a feasible and consistent potential function  $\pi_f$ , a node  $u$  is settled by increasing value of  $d(s, u, \tau_0) + \pi_f(u)$ . Hence, for any unsettled node  $w$ , we have  $d(s, v, \tau_0) + \pi_f(v) \leq d(s, w, \tau_0) + \pi_f(w)$ . ■

Let  $v'$  be any node settled by the forward search, and let  $w$  be a node that has not been settled. Proposition 7.3 suggests that we can use

$$\pi_b^*(w) = \max\{\pi_b(w), d(s, v', \tau_0) + \pi_f(v') - \pi_f(w)\} \quad (4)$$

as a lower bound to  $d(s, w, \tau_0)$  during the backward search. To maximize this bound, we should choose  $v'$  that maximizes  $d(s, v', \tau_0) + \pi_f(v')$ , that is, choose  $v'$  as the last node settled by the forward search. We remark that  $\pi_b^*$  is *not* a feasible potential for the backward search: it yields valid lower bounds for the time-dependent graph, but it could overestimate distances on  $G_\lambda$ . The following lemma gives a correct way to use (4).

**Lemma 7.4.** *If the node  $v'$  used to compute the potential function  $\pi_b^*$  defined by (4) is fixed, then  $\pi_b^*(v) \leq \pi_b^*(u) +$*

$\lambda(u, v)$  for each arc  $(u, v) \in A$  such that  $v$  has not been settled by the forward search.

**Proof.** By definition,  $\pi_b^*(v) = \max\{\pi_b(v), \alpha - \pi_f(v)\}$ , where with  $\alpha$  we denoted the key of  $v'$ , that is,  $d(s, v', \tau_0) + \pi_f(v')$ , which is fixed by hypothesis. Consider the case  $\pi_b^*(v) = \pi_b(v)$ ; then, because the landmark potential functions  $\pi_b$  and  $\pi_f$  are consistent,  $\pi_b^*(v) = \pi_b(v) \leq \pi_b(u) + \lambda(u, v) \leq \pi_b^*(u) + \lambda(u, v)$ . Now consider the case  $\pi_b^* = \alpha - \pi_f(v)$ ; then  $\pi_b^*(v) = \alpha - \pi_f(v) \leq \alpha - \pi_f(u) + \lambda(u, v) \leq \pi_b^*(u) + \lambda(u, v)$ , which completes the proof. ■

One could think that feasibility and consistency of  $\pi_b^*$  for the backward search on  $G_\lambda$  would follow from Lemma 7.4 by chaining the inequalities  $\pi_b^*(v) \leq \pi_b^*(u) + \lambda(u, v)$  for nodes on the shortest path to  $s$ . However, this is not the case because Lemma 7.4 is only valid for nodes that have not been settled by the forward search; therefore, the chain of inequalities would fail as soon as we encounter a node that has been settled by the forward search. We can still prove correctness of our algorithm with tightened bounds, thanks to Theorem 7.1.

**Theorem 7.5.** *Algorithm 1 is correct if we use (4) for fixed  $v'$  as potential function for the backward search.*

**Proof.** Let  $d_b(u)$  be the distance from a node  $u$  to node  $t$  computed by the backward search. We will prove that, when a node  $u$  on the shortest path from  $s$  to  $t$  is settled by the backward search,  $d_b(u) \leq d(u, t, d(s, u, \tau_0)) \forall \tau_0 \in T$ . By Proposition 7.3 and Theorem 7.1, this is enough to prove our statement.

Let  $q^* = (v_1 = u, \dots, v_n = t)$  be the shortest path from  $u$  to  $t$  on  $G_\lambda$ . We proceed by induction on  $i : n, \dots, 1$  to prove that each node  $v_i$  is settled with the correct distance on  $G_\lambda$ , i.e.,  $d_b(v_i) = d_\lambda(v_i, t)$ . It is trivial to see that the nodes  $v_n$  and  $v_{n-1}$  are settled with the correct distance on  $G_\lambda$ . For the induction step, suppose  $v_i$  is settled with the correct distance  $d_b(v_i) = d_\lambda(v_i, t)$ . By Lemma 7.4, we have  $d_b(v_i) + \pi_b^*(v_i) \leq d_b(v_i) + \lambda(v_{i-1}, v_i) + \pi_b^*(v_{i-1}) = d_\lambda(v_{i-1}, t) + \pi_b^*(v_{i-1}) \leq d_b(v_{i-1}) + \pi_b^*(v_{i-1})$ ; hence,  $v_i$  is extracted from the queue before  $v_{i-1}$ . This means that  $v_{i-1}$  will be settled with the correct distance  $d_b(v_{i-1}) = d_\lambda(v_{i-1}, t)$ , and the induction step is proven.

Thus,  $u$  will be settled with distance  $d_b(u) = d_\lambda(u, t) \leq d(u, t, d(s, u, \tau_0))$ , which proves our statement. ■

By Theorem 7.5, Algorithm 1 is correct when using  $\pi_b^*$  only if we assume that the node  $v'$  used in (4) is fixed at each backward search iteration. Thus, we do the following: we set up 10 checkpoints during the query; when a checkpoint is reached, the node  $v'$  used to compute (4) is updated, and the backward search queue is flushed and filled again using the updated  $\pi_b^*$ . This is enough to guarantee correctness. The checkpoints are computed comparing the initial lower bound  $\Delta = \pi_f(t)$  and the current distance from the source node, both for the forward search: the initial lower bound is divided by

10 and, whenever the current distance from the source node exceeds  $k\Delta/10$  with  $k \in \{1, \dots, 10\}$ ,  $\pi_b^*$  is updated.

## 8. COMPUTATIONAL RESULTS

In this section, we present an extensive experimental evaluation of the TDALT algorithm. Our implementation is written in C++ using the Standard Template Library. For the priority queue, we use a binary heap. Other types of priority queues were also tested. In our computational experience, the impact of the choice of the priority queue has almost no influence on the performance of speed-up techniques. Our tests were executed on one core of an AMD Opteron 2218 running SUSE Linux 10.3. The machine is clocked at 2.6 GHz, has 16 GB of RAM and  $2 \times 1$  MB of L2 cache. The program was compiled with GCC 4.2.1, using optimization level 3.

Unless otherwise stated, we use 16 maxCover landmarks (see Section 3), computed on the input graph using the lower bounding function  $\lambda$  to weight arcs, and we use (4) as potential function for the backward search, with 10 checkpoints (see Section 7). We use dynamic landmark selection, as suggested in [22]. When performing random  $s$ - $t$  queries, the source  $s$ , target  $t$ , and the starting time  $\tau_0$  are picked uniformly at random and results are based on 10,000 queries. We evaluate the query performance by reporting the average number of settled nodes, that is, the number of nodes extracted from the priority queues, the number of relaxed arcs, and the resulting running times.

### 8.1. Inputs

We tested our algorithm on two different road networks: the road network of Western Europe, which has  $\sim 18$  million vertices and 42.6 million arcs, and the road network of Germany, which has 4.7 million nodes and 10.8 million arcs.

Our German data contains five different realistic traffic scenarios, generated from traffic simulations: Monday, midweek (Tuesday till Thursday), Friday, Saturday, and Sunday. As expected, congestion of roads is higher during the week than on the weekend:  $\approx 8\%$  of arcs are time-dependent for Monday, midweek, and Friday. The corresponding figures for Saturday and Sunday are  $\approx 5\%$  and  $\approx 3\%$ , respectively. All data has been provided by PTV AG for scientific use.

Unfortunately, our European data set does not contain traffic data. We therefore used artificially generated costs. To model the time-dependent costs on each arc, we developed a heuristic algorithm, based on statistics gathered using real-world data on a limited-size road network; we used piecewise linear cost functions, with one breakpoint for each hour over a day. Arc costs are generated assigning, at each node, several random values that represent peak hour (i.e., hour with maximum traffic), duration and speed of traffic increase/decrease for a traffic jam; for each node, two traffic jams are generated, one in the morning and one in the afternoon. Then, for each arc in a node's arc star, a speed profile is generated, using the traffic jam's characteristics of the corresponding node, and assigning a random increase

TABLE 1. Performance of the time-dependent versions of Dijkstra, unidirectional ALT, and our bidirectional approach.

| Method   | $K$  | Error    |          |         | Query           |           |           |           |
|----------|------|----------|----------|---------|-----------------|-----------|-----------|-----------|
|          |      | Rate (%) | Relative |         | # Settled nodes |           |           | Time (Ms) |
|          |      |          | Avg (%)  | Max (%) | Phase 1         | Phase 2   | Phase 3   |           |
| Dijkstra | —    | 0.0      | 0.000    | 0.00    | —               | —         | 8 877 158 | 5 757.4   |
| uni-ALT  | —    | 0.0      | 0.000    | 0.00    | —               | —         | 2 143 160 | 1 520.8   |
| TDALT    | 1.00 | 0.0      | 0.000    | 0.00    | 132 129         | 2 556 840 | 3 009 320 | 2 842.0   |
|          | 1.05 | 3.1      | 0.012    | 3.91    | 132 129         | 1 244 050 | 1 574 750 | 1 379.2   |
|          | 1.07 | 6.6      | 0.034    | 6.06    | 132 129         | 849 171   | 1 098 470 | 915.4     |
|          | 1.10 | 18.1     | 0.106    | 7.79    | 132 129         | 473 414   | 622 466   | 481.9     |
|          | 1.12 | 26.1     | 0.182    | 10.57   | 132 129         | 337 353   | 444 991   | 325.0     |
|          | 1.15 | 35.4     | 0.292    | 10.57   | 132 129         | 236 108   | 311 209   | 214.2     |
|          | 1.20 | 43.0     | 0.485    | 19.40   | 132 129         | 171 154   | 225 557   | 145.3     |
|          | 1.25 | 45.4     | 0.589    | 21.64   | 132 129         | 148 856   | 196 581   | 122.3     |
|          | 1.30 | 46.4     | 0.656    | 21.64   | 132 129         | 139 089   | 184 143   | 111.6     |
|          | 1.35 | 47.0     | 0.704    | 21.64   | 132 129         | 134 582   | 178 410   | 107.4     |
|          | 1.50 | 47.1     | 0.722    | 21.64   | 132 129         | 132 299   | 175 468   | 105.4     |
|          | 1.75 | 47.2     | 0.726    | 30.49   | 132 129         | 132 131   | 175 248   | 105.4     |
|          | 2.00 | 47.2     | 0.726    | 30.49   | 132 129         | 132 130   | 175 247   | 105.4     |

factor between 1.5 and 3 to represent that arc’s slowdown during peak hours with respect to uncongested hours. We do not assign speed profile to arcs that have both endpoints at nodes with level 0 in a preconstructed Highway Hierarchy [32], because they represent “unimportant” nodes of the road network (i.e., nodes that only appear in local shortest paths, as opposed to long-distance shortest paths). As a result, those arcs will have the same travelling time value throughout the day; for all other arcs, we use the traffic jam values associated with the endpoint with smallest ID.

This method was developed to ensure spatial coherency between traffic increases, that is, if a certain arc is congested at a given time, then it is likely that adjacent arcs will be congested too. This is a basic principle of traffic analysis [27].

## 8.2. Random Queries

Table 1 reports the results of our bidirectional ALT variant on time-dependent networks for different approximation values  $K$  using the European road network as input. Pre-processing takes approximately 75 min and produces 128 additional bytes per node. The extra space is required because for each node we store distances to and from all landmarks.

As the performed TDALT queries compute approximated results instead of optimal solutions, we record three different statistics to characterize the solution quality: error rate, average relative error, maximum relative error. By error rate, we mean the percentage of computed suboptimal paths over the total number of queries. By relative error on a particular query, we mean the relative percentage increase of the approximated solution over the optimum, computed as  $\omega/\omega^* - 1$ , where  $\omega$  is the cost of the approximated solution computed by our algorithm and  $\omega^*$  is the cost of the optimum computed by Dijkstra’s algorithm. We report average and maximum values of this quantity over the set of all

queries. We also report the number of nodes settled at the end of each phase of our algorithm, denoting them with the labels Phase 1, Phase 2, and Phase 3.

As expected, we observe a clear trade-off between the quality of the computed solution and query performance. For an approximation factor of  $K = 2.0$ , on the European road network queries are on average 55 times faster than Dijkstra’s algorithm, but almost 50% of the computed paths will be suboptimal. In our tests, the average relative error is small, even though in the worst case the approximated solution has a cost which is 50% larger than the optimal value. One reason for this poor solution quality is that for such high approximation values, Phase 2 is very short. As a consequence, nodes in the middle of the shortest path are not explored by our approach, and the meeting point of the two search scopes is far from being the optimal one. By decreasing the value of the approximation constant  $K$ , we are able to obtain solutions that are very close to the optimum, and performance is significantly better than for unidirectional ALT or Dijkstra. In our experiments, a very good trade-off between average quality and performance is achieved with an approximation value of  $K = 1.15$ , which yields average query times smaller than 215 ms with an average error of 0.3% and a maximum recorded relative error of 10.6%. In road networks, the speed profiles are not completely accurate; therefore, a slightly suboptimal solution (on average, the cost increases by less than 0.3% over the optimum for  $K = 1.15$ ) is usually an acceptable solution. By decreasing  $K$  to values  $< 1.05$ , it does not pay off to use the bidirectional variant any more, as the unidirectional variant of ALT is faster and is always correct.

An interesting observation is that for  $K = 2.0$ , switching from a static to a time-dependent scenario increases query times only by a factor of  $\approx 2$ : on the European road network, in a static scenario, ALT has query times of 53.6 ms (see Ref. [13]), whereas our time-dependent variant yields query

TABLE 2. Performance of the time-dependent versions of Dijkstra, unidirectional ALT, and our bidirectional approach without the tightened potential function  $\pi_b^*$  defined as in (4).

| Method   | $K$  | Error    |          |         | Query         |           |           |           |
|----------|------|----------|----------|---------|---------------|-----------|-----------|-----------|
|          |      | Rate (%) | Relative |         | Settled nodes |           |           | Time (ms) |
|          |      |          | Avg (%)  | Max (%) | Phase 1       | Phase 2   | Phase 3   |           |
| Dijkstra | —    | 0.0      | 0.000    | 0.00    | —             | —         | 8 877 158 | 5 757.4   |
| uni-ALT  | —    | 0.0      | 0.000    | 0.00    | —             | —         | 2 143 160 | 1 520.8   |
| TDALT    | 1.00 | 0.0      | 0.000    | 0.00    | 719 650       | 3 763 990 | 3 862 070 | 3 291.6   |
|          | 1.05 | 3.5      | 0.023    | 4.88    | 719 650       | 2 996 940 | 3 238 120 | 2 683.5   |
|          | 1.07 | 5.5      | 0.046    | 6.94    | 719 650       | 2 519 750 | 2 874 500 | 2 290.7   |
|          | 1.10 | 12.1     | 0.123    | 9.45    | 719 650       | 1 810 340 | 2 201 870 | 1 619.2   |
|          | 1.12 | 20.1     | 0.237    | 10.93   | 719 650       | 1 416 240 | 1 772 080 | 1 218.4   |
|          | 1.15 | 32.1     | 0.474    | 14.35   | 719 650       | 1 049 750 | 1 345 930 | 842.0     |
|          | 1.20 | 44.4     | 0.788    | 19.42   | 719 650       | 824 331   | 1 079 290 | 618.3     |
|          | 1.25 | 50.5     | 0.994    | 24.57   | 719 650       | 755 262   | 996 631   | 553.3     |
|          | 1.30 | 53.3     | 1.104    | 24.57   | 719 650       | 735 524   | 972 294   | 531.5     |
|          | 1.35 | 54.7     | 1.166    | 24.57   | 719 650       | 727 843   | 962 950   | 526.5     |
|          | 1.50 | 56.1     | 1.248    | 28.16   | 719 650       | 720 359   | 953 704   | 524.7     |
|          | 1.75 | 56.3     | 1.261    | 39.34   | 719 650       | 719 661   | 952 947   | 519.0     |
|          | 2.00 | 56.4     | 1.262    | 39.41   | 719 650       | 719 650   | 952 933   | 518.2     |

times of 105 ms. We also note that for our bidirectional search there is an additional overhead that increases the time spent per node with respect to unidirectional ALT: on the European road network, using an approximation factor of  $K = 1.05$  yields similar query times to unidirectional ALT, but the number of nodes settled by the bidirectional approach is almost 30% smaller. We conjecture that this is due to the following facts: in the bidirectional approach, one has to check at each iteration if the current node has been settled in the opposite direction. During Phase 2, the upper bound has to be updated from time to time. The cost of these operations, added to the phase-switch checks, possibly accounts for the increase of running time per settled node.

We also report the results obtained on the European road network using the unmodified ALT potential function  $\pi_b$  for the backward search, instead of the tightened one  $\pi_b^*$  defined as in (4). These can be found in Table 2, which has the same column labels as Table 1. Comparing query times with the same value of the approximation constant  $K$ , we see that using the potential function  $\pi_b^*$  yields a significant improvement over  $\pi_b$ . The difference in performance is larger as  $K$  increases. For  $K = 1$  the difference is very small; for  $K = 1.05$ , the algorithm with  $\pi_b$  is 95% slower than the one with  $\pi_b^*$ , and the slowdown increases to 236% for  $K = 1.10$  and to 293% for  $K = 1.15$ . With the largest approximation factor that we tested in our experiments,  $K = 2$ , the algorithm without the tightened potential function is more than five times slower. The same behavior is observed in terms of the number of settled nodes. For  $K = 1$ , the number is very similar (only a 28% increase when not using  $\pi_b^*$ ). The ratio rapidly grows until it reaches a 444% increase for  $K = 2$ . Thus, a great deal of the computational improvement that the bidirectional variant obtains over Dijkstra’s algorithm and unidirectional ALT is due to the use of tightened bounds. If we use the standard ALT potential

function  $\pi_b$  for the backward search, then we do not manage to obtain a speed-up of more than a factor 3 with respect to unidirectional ALT, and this comes at the price of sub-optimal solutions. Summarizing, our bidirectional approach has the great advantage of deriving better lower bounds for the time-dependent search when compared with the original ALT bounds. The new potential function leads to large computational improvement.

### 8.3. Local Queries

For random queries on the European road network, our bidirectional TDALT algorithm with  $K = 1.15$  is roughly 6.5 times faster than unidirectional ALT on average. To gain insight whether this speed-up derives from small- or large-range queries, Figure 4 reports the query times with respect to the Dijkstra rank.<sup>1</sup> These values were gathered on the European road network instance. Note that we use a logarithmic scale due to the fluctuating query times of bidirectional TDALT. Comparing both ALT versions, we observe that switching from uni- to bidirectional queries pays off especially for long-distance queries. This is not surprising. For small distances, the overhead for bidirectional routing is not counterbalanced by a significant decrease in the number of explored nodes. Hence, unidirectional ALT is faster for local queries. For ranks of  $2^{24}$ , the median of the bidirectional variant is almost two orders of magnitude lower than for the unidirectional variant. Another interesting observation is the fact that some outliers of bidirectional TDALT are almost as slow as the unidirectional variant. Comparing different approximation values, we observe that query times differ by roughly the same factor for all ranks less than  $2^{23}$ .

<sup>1</sup>For an  $s-t$  query, the Dijkstra rank of node  $t$  is the number of nodes settled before  $t$  is settled. Thus, it is some kind of distance measure.

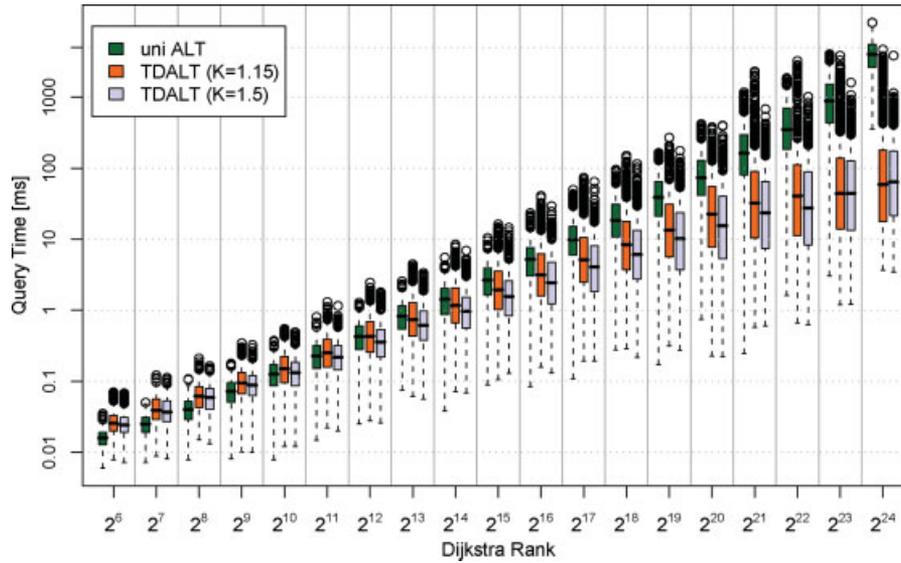


FIG. 4. Comparison of unidirectional ALT and bidirectional TDALT using the Dijkstra rank methodology [32]. The results are represented as box-and-whisker plot: each box spreads from the lower to the upper quartile and contains the median, the whiskers extend to the minimum and maximum value omitting outliers, which are plotted individually. [Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

#### 8.4. Number of Landmarks

In static scenarios, query times of bidirectional TDALT can be significantly reduced by increasing the number of landmarks to 32 or even 64 (see [13]). To check whether this also holds for our time-dependent variant, we recorded our algorithm's performance using different numbers of landmarks. Table 3 reports those results on the European road network. We evaluate eight maxCover landmarks (yielding a preprocessing effort of 33 min and an overhead of 64 bytes per node), 16 maxCover landmarks (75 minutes, 128 bytes per node) and 32 Avoid landmarks (29 min, 256 bytes per node). In the latter case, we used Avoid instead of maxCover because obtaining 32 landmarks with maxCover required too much computing time. We do not report error rates here, because the number of landmarks has almost no impact on the quality of the computed paths. Surprisingly, the number of landmarks has a very small influence on the performance of

TDALT. In fact, increasing the number of landmarks yields larger average query times for unidirectional ALT and for bidirectional TDALT with low  $K$ -values and, thus, makes the algorithm less efficient. This is due the fact that the search space decreases only slightly, but the additional overhead for accessing landmarks increases. However, for larger values  $K$ , more landmarks results in faster query times. With  $K = 2.0$  and 32 landmarks, we are able to perform time-dependent queries 70 times faster than plain Dijkstra, but the solution quality in this case is no better than in the 16 landmarks case. Summarizing, for  $K > 1.10$  increasing the number of landmarks has a positive effect on computational times, although switching from 16 to 32 landmarks does not yield the same benefits as from 8 to 16, and thus in our experiments is not worth the extra memory. On the other hand, for  $K \leq 1.10$  and for unidirectional ALT, increasing the number of landmarks has a negative effect on computational times and, thus, is never a good choice in our experiments.

TABLE 3. Performance of unidirectional ALT and bidirectional TDALT with different number of landmarks in a time-dependent scenario.

|         | K    | 8 landmarks |           | 16 landmarks |           | 32 landmarks |           |
|---------|------|-------------|-----------|--------------|-----------|--------------|-----------|
|         |      | # Settled   | Time (ms) | # Settled    | Time (ms) | # Settled    | Time (ms) |
| uni-ALT | —    | 2 280 420   | 1 446.4   | 2 143 160    | 1 520.8   | 2 056 190    | 1 623.3   |
| TDALT   | 1.00 | 3 147 440   | 2 745.5   | 3 009 320    | 2 842.0   | 2 931 080    | 2 953.3   |
|         | 1.05 | 1 714 210   | 1 373.8   | 1 574 750    | 1 379.2   | 1 516 710    | 1 409.5   |
|         | 1.10 | 768 368     | 540.2     | 622 466      | 481.9     | 561 253      | 464.2     |
|         | 1.15 | 461 259     | 293.5     | 311 209      | 214.2     | 250 248      | 184.4     |
|         | 1.20 | 375 900     | 230.6     | 225 557      | 145.3     | 164 419      | 111.1     |
|         | 1.50 | 326 076     | 195.8     | 175 468      | 105.3     | 113 040      | 68.1      |
|         | 2.00 | 325 801     | 195.8     | 175 247      | 105.4     | 112 826      | 68.0      |

TABLE 4. Performance of TDALT on the German road network instance.

| Scenario | Algorithm | K    | Error    |          |         | Query          |               |           |
|----------|-----------|------|----------|----------|---------|----------------|---------------|-----------|
|          |           |      | Rate (%) | Relative |         | #Settled Nodes | #Relaxed Arcs | Time (ms) |
|          |           |      |          | Avg (%)  | Max (%) |                |               |           |
| Monday   | uni-ALT   | —    | 0.0      | 0.000    | 0.00    | 193 087        | 230 665       | 140.38    |
|          | TDALT     | 1.00 | 0.0      | 0.000    | 0.00    | 106 743        | 127 190       | 88.53     |
|          |           | 1.15 | 12.5     | 0.094    | 13.02   | 51 137         | 60 838        | 37.23     |
|          |           | 1.50 | 12.5     | 0.096    | 24.27   | 51 119         | 60 816        | 37.12     |
| Midweek  | uni-ALT   | —    | 0.0      | 0.000    | 0.00    | 200 236        | 239 112       | 147.20    |
|          | TDALT     | 1.00 | 0.0      | 0.000    | 0.00    | 116 476        | 138 696       | 98.27     |
|          |           | 1.15 | 12.4     | 0.094    | 14.32   | 50 764         | 60 398        | 36.91     |
|          |           | 1.50 | 12.5     | 0.097    | 27.59   | 50 742         | 60 371        | 36.86     |
| Friday   | uni-ALT   | —    | 0.0      | 0.000    | 0.00    | 196 551        | 235 083       | 143.52    |
|          | TDALT     | 1.00 | 0.0      | 0.000    | 0.00    | 116 857        | 139 175       | 98.28     |
|          |           | 1.15 | 12.0     | 0.096    | 14.03   | 50 891         | 60 550        | 36.92     |
|          |           | 1.50 | 12.1     | 0.098    | 30.77   | 50 874         | 60 531        | 36.82     |
| Saturday | uni-ALT   | —    | 0.0      | 0.000    | 0.00    | 148 331        | 177 568       | 100.07    |
|          | TDALT     | 1.00 | 0.0      | 0.000    | 0.00    | 63 717         | 76 001        | 47.41     |
|          |           | 1.15 | 10.5     | 0.088    | 13.97   | 50 042         | 59 607        | 36.00     |
|          |           | 1.50 | 10.6     | 0.089    | 26.17   | 50 036         | 59 600        | 35.63     |
| Sunday   | uni-ALT   | —    | 0.0      | 0.000    | 0.00    | 142 631        | 170 670       | 92.79     |
|          | TDALT     | 1.00 | 0.0      | 0.000    | 0.00    | 58 956         | 70 333        | 42.96     |
|          |           | 1.15 | 10.4     | 0.088    | 14.28   | 50 349         | 59 994        | 36.04     |
|          |           | 1.50 | 10.5     | 0.089    | 32.08   | 50 345         | 59 988        | 35.74     |

Scenario depicts the degree of perturbation.

### 8.5. Traffic Days

Next, we focus on the impact of arc cost perturbation on TDALT, where by perturbation we mean the difference between the static lower bounds used to compute landmark distances and the time-dependent costs. Table 4 reports the performance of unidirectional and bidirectional time-dependent ALT for different traffic days on the German road network. Dijkstra settles 2.2 million nodes in  $\approx 1.5$  s in this setup, independent of the traffic day.

The degree of perturbation has only a mild impact on unidirectional ALT and bidirectional TDALT. In a low traffic scenario, unidirectional ALT queries are up 16 times faster than plain Dijkstra, while this value drops to 10 if more arcs are perturbed. Switching from exact to approximate queries does not pay off in low traffic scenarios. The gain in performance is only around 20%, which seems rather low compared with the loss in quality of paths. However, this value increases to a factor of up to 3 in high traffic scenarios. Still, comparing Tables 1 and 4, the gain in performance for dropping correctness is much lower for Germany than for Europe. Possibly, this derives from the size of the graph. With increasing graph size, lower bounds get worse as the gap between lower bound distance and time-dependent distance increases. This would also explain why speed-ups for unidirectional ALT are higher for Germany than for Europe.

## 9. CONCLUSIONS

We have presented an algorithm that applies bidirectional search on a time-dependent road network, where the

backward search is used to bound the set of nodes that have to be explored by the forward search; this algorithm is based on the ALT variant of the  $A^*$  algorithm. We have discussed related theoretical issues, and we proved the algorithm's correctness. Our idea can be adapted and applied to several shortest path algorithms and lays the foundations for future work. Extensive computational experiments show that this algorithm is very effective in practice if we are willing to accept a small approximation factor. The exact version of our algorithm is slower than unidirectional ALT, but a nearly optimal variant of our algorithm is several times faster. For practical applications, this is usually a good compromise.

## REFERENCES

- [1] R. Ahuja, T. Magnanti, and J. Orlin, *Network flows: theory, algorithms, and applications*, Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1993.
- [2] H. Bast, S. Funke, P. Sanders, and D. Schultes, Fast routing in road networks with transit nodes, *Science* 316 (2007), 566.
- [3] G.V. Batz, D. Delling, P. Sanders, and C. Vetter, "Time-Dependent Contraction Hierarchies," *Proc 11th Workshop Algorithm Eng Exp (ALENEX'09)*, SIAM, 2009, pp. 97–105.
- [4] R. Bauer and D. Delling, SHARC: Fast and robust unidirectional routing, *ACM J Exp Algorithmics* 14 (2009), 2.4.
- [5] L. Buriol, M. Resende, and M. Thorup, Speeding up dynamic shortest path algorithms, *INFORMS J Comput* 20 (2008), 191–204.

- [6] I. Chabini, Discrete dynamic shortest path problems in transportation applications: complexity and algorithms with optimal run time, *Transport Res Rec* 1645 (1998), 170–175.
- [7] I. Chabini and S. Lan, Adaptations of the  $A^*$  algorithm for the computation of fastest paths in deterministic discrete-time dynamic networks, *IEEE Trans Intell Transport Syst* 3 (2002), 60–74.
- [8] K. Cooke and E. Halsey, The shortest route through a network with time-dependent internodal transit times, *J Math Anal Appl* 14 (1966), 493–498.
- [9] C. Daganzo, Reversibility of time-dependent shortest path problem, Technical report, Institute of Transportation Studies, University of California, UCB-ITS-RR-98-14, 1998, Berkeley.
- [10] D. Delling, Time-Dependent SHARC-Routing, *Algorithmica* 60 (2011), 60–94, Special Issue: European Symposium on Algorithms 2008.
- [11] D. Delling and G. Nannicini, “Bidirectional core-based routing in dynamic time-dependent road networks,” *Proceedings of the 19th International Symposium on Algorithms and Computation (ISAAC 08)*, Lecture Notes in Computer Science, S.-H. Hong, H. Nagamochi, and T. Fukunaga (Editors), Vol. 5369, Springer, Berlin, 2008, pp. 813–824.
- [12] D. Delling, P. Sanders, D. Schultes, and D. Wagner, “Engineering route planning algorithms,” *Algorithmics of Large and Complex Networks*, Lecture Notes in Computer Science, J. Lerner, D. Wagner, and K.A. Zweig (Editors), Vol. 5515, Springer, 2009, pp. 117–139.
- [13] D. Delling and D. Wagner, “Landmark-based routing in dynamic graphs,” 6th Workshop on Experimental Algorithms, LNCS, C. Demetrescu (Editor), Vol. 4525, Springer, New York, 2007, pp. 52–65.
- [14] D. Delling and D. Wagner, “Time-Dependent Route Planning,” *Robust and Online Large-Scale Optimization*, Lecture Notes in Computer Science, R.K. Ahuja, R.H. Möhring, and C. Zaroliagis (Editors), Vol. 5868, Springer, 2009, pp. 207–230.
- [15] E. Dijkstra, A note on two problems in connexion with graphs, *Numerische Mathematik* 1 (1959), 269–271.
- [16] S. Dreyfus, An appraisal of some shortest-path algorithms, *Operations Res* 17 (1969), 395–412.
- [17] L.R. Ford and D.R. Fulkerson, *Modern Heuristic Techniques for Combinatorial Problems*, Princeton University Press, Princeton, NJ, 1962.
- [18] M. Fredman and R. Tarjan, Fibonacci heaps and their use in improved network optimization algorithms, *J ACM* 34 (1987), 596–615.
- [19] R. Geisberger, P. Sanders, D. Schultes, and D. Delling, “Contraction Hierarchies: Faster and simpler hierarchical routing in road networks,” *Proceedings of the 7th Workshop on Experimental Algorithms (WEA’08)*, Lecture Notes in Computer Science, C.C. McGeoch (Editor), Vol. 5038, Springer, 2008, pp. 319–333.
- [20] A. Goldberg and C. Harrelson, “Computing the shortest path:  $A^*$  meets graph theory,” *Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2005)*, SIAM, Philadelphia, 2005, pp. 156–165.
- [21] A. Goldberg, H. Kaplan, and R. Werneck, “Reach for  $A^*$ : Efficient point-to-point shortest path algorithms,” *Proceedings of the 8th Workshop on Algorithm Engineering and Experiments (ALENEX 06)*, Lecture Notes in Computer Science, Springer, 2006, pp. 129–143.
- [22] A. Goldberg and R. Werneck, “Computing point-to-point shortest paths from external memory,” *Proceedings of the 7th Workshop on Algorithm Engineering and Experimentation (ALENEX 05)*, C. Demetrescu, R. Sedgewick, and R. Tamassia (Editors), SIAM, Philadelphia, 2005, pp. 26–40.
- [23] B.L. Golden and M. Ball, Shortest paths with Euclidean distances: An explanatory model, *Networks* 8 (1978), 297–314.
- [24] E. Hart, N. Nilsson, and B. Raphael, A formal basis for the heuristic determination of minimum cost paths, *IEEE Trans Sys Sci Cybernetics SSC-4* (1968), 100–107.
- [25] T. Ikeda, M. Tsu, H. Imai, S. Nishimura, H. Shimoura, T. Hashimoto, K. Tenmoku, and K. Mitoh, “A fast algorithm for finding better routes by AI search techniques,” *Proc IEEE Vehicle Navigation and Information Systems Conference*, 2004, pp. 291–296.
- [26] D.E. Kaufman and R.L. Smith, Fastest paths in time-dependent networks for intelligent vehicle-highway systems application, *J Intell Transport Syst* 1 (1993), 1–11.
- [27] B.S. Kerner, *The physics of traffic*, Springer, Berlin, 2004.
- [28] G. Nannicini, P. Baptiste, G. Barbier, D. Kroh, and L. Liberti, Fast paths in large-scale dynamic road networks, *Comput Opt Appl* 45 (2010), 143–158.
- [29] G. Nannicini, D. Delling, L. Liberti, and D. Schultes, “Bidirectional  $A^*$  search for time-dependent fast paths,” *Proceedings of the 8th Workshop on Experimental Algorithms (WEA 2008)*, Lecture Notes in Computer Science, C. McGeoch (Editor), Vol. 5038, Springer, New York, 2008, pp. 334–346.
- [30] G. Nannicini and L. Liberti, Shortest paths on dynamic graphs, *Int Trans Oper Res* 15 (2008), 551–563.
- [31] A. Orda and R. Rom, Shortest-path and minimum delay algorithms in networks with time-dependent edge-length, *J ACM* 37 (1990), 607–625.
- [32] P. Sanders and D. Schultes, “Highway hierarchies hasten exact shortest path queries,” *13th Annual European Symposium on Algorithms (ESA 2005)*, Lecture Notes in Computer Science, G. Støltting Brodal and S. Leonardi (Editors), Vol. 3669, Springer, 2005, pp. 568–579.
- [33] P. Sanders and D. Schultes, “Dynamic highway-node routing,” *6th Workshop on Experimental Algorithms, LNCS, C. Demetrescu (Editor)*, Vol. 4525, Springer, New York, 2007, pp. 66–79.
- [34] R. Sedgewick and J. Vitter, Shortest paths in Euclidean graphs, *Algorithmica* 1 (1986), 31–48.
- [35] D. Wagner, T. Willhalm, and C. Zaroliagis, Geometric containers for efficient shortest-path computation, *ACM J Exp Algorithmics* 10 (2005), 1–30.