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PREFERENTIAL POLARIZATION MEASURES

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Abstract

Given opinions of members of a society on a set of policies, as ordinal preferences; an approach to polarization is introduced. The concept here is considering polarization in a society as an aggregation of pairwise antagonisms, which depend on identification within groups as well as alienation among groups. Among measures which comply to this sort of conceptualization, a class of functions which satisfies certain plausible properties is introduced for the case of three alternatives. This class coincides with the class characterized for unidimensional spaces in Esteban and Ray (1994).
Highlights:

- We model how to measure polarization with ordinal preferences.
- Polarization is seen as the aggregation of pairwise antagonisms.
- We propose anonymous and neutral measures that satisfy certain plausible properties.

Keywords: polarization measures, ordinal preferences, alienation, identification

1. Introduction and the Model

Since polarization is understood to be closely related to social discord and induction of enmity, it is of interest how to understand and conceptualize polarization within a society with ordinal preferences. One approach, not specifically preference based, suggests that it should be considered as the aggregation of pairwise antagonisms within society. In this paper, following Esteban and Ray [6], we suggest that antagonism felt by an individual towards another depends on the alienation between the two and also on the feeling of identification he/she enjoys by being in wherever he/she is. Alienation then can be thought of as a function of the dissimilarity between the two stances. We further postulate that identification can be seen as a function of the support of one's opinion. In a formal way, this would mean to measure polarization in a society with a function as follows.

$$\sum_j \sum_i a(I(m_i), A(d(o_i, o_j)))$$

Here the function $I(m_i)$ represents how much i is identified with his/her position, as a function of m_i, the share of the population who thinks the same with i. The function $A(d(o_i, o_j))$ represents the alienation as an increasing function of the distance of opinions of i and j. Finally, the function $a(I, A)$ represents the antagonism felt by i towards j, as a function of identification i enjoys by being wherever i is and alienation between i and j. Summing up for all directed pairs is then a measure of polarization as discussed. We construe the set of properties in Esteban and Ray [6] and show that a subclass of the functions above satisfies those properties for ordinal preferences over three alternatives. To our knowledge, this is the first attempt at this question.[1]

Let A be the set of alternatives with $|A| = m$. There are $m!$ possible linear orders and the set of all those is denoted $\mathcal{L}(A)$. Let $\bar{P} = (P^1, P^2, ..., P^{m!})$ be an arbitrary listing of the set $\mathcal{L}(A)$. A preference profile for a society of N individuals is a member of $\mathcal{L}(A)^N$. Given $\bar{P}$ and $P_N \in \mathcal{L}(A)^N$ while letting $N_i^{\bar{P}} = |\{j \in N : P_j = P^i\}|$ we have a representing preference distribution $(\pi, \bar{P})$ where $\pi = (\pi_1, \pi_2, ..., \pi_{m!})$ is such that $\pi_i = N_i^{\bar{P}}/N$. Hence we have that $\sum_{i=1}^{m!} \pi_i = 1$.

A polarization measure is a mapping $\mathcal{P} : \mathcal{L}(A)^N \to \mathbb{R}_+$. Denoting the set of all preference distributions $(\pi, \bar{P})$ by $\mathcal{D}$, we have an equivalent depiction of polarization measures; $\mathcal{P} : \mathcal{D} \to \mathbb{R}_+$.

In the following analysis, we will be employing a particular metric on linearly ordered preferences, namely *Kemeny metric*.[2] Given any $P, P' \in \mathcal{L}(A)$, the Kemeny distance between P and P' is defined as $d_\kappa(P, P') = |P \backslash P'|$ where $P \backslash P'$ is the symmetric difference. The graph for three alternatives is as follows.

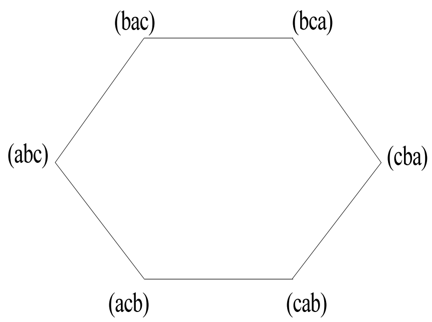

Figure 1: Kemeny graph, three alternatives

Now we introduce a set of properties on preferential polarization measures following the axiomatization in Esteban and Ray [6].[3]

Definition 1 (Property 1). *Let $\pi_i > \pi_j = \pi_k$ be the only masses of a profile such that π_j and π_k are at least as close to each other as they are to π_i. A polarization measure is said to satisfy* Property 1 *if joining the two smaller masses at a point at least as further as the average distance to π_i increases polarization.*

[1] Baldiga and Green [2] analyze social choice functions in terms of assent maximizing attributes. Alcalde-Unzu and Vorsatz [1] provide ways to measure consensus where their analysis which pertains to a consequentialist approach is not compatible with framework of this paper.

[2] This widely known and used metric is due to Kemeny [7] for which Bogart [3] and Kemeny and Snell [8], *inter alia*, provide axiomatic characterizations. Can and Storcken [5] provides a refinement of previous characterizations. For a general treat of weighted distances between preferences, see Can [4].

[3] In fact, *Property i* here may be matched with *Axiom i* in Esteban and Ray [6], although we keep in mind that there is no unique translation of each axiom into the current setting.

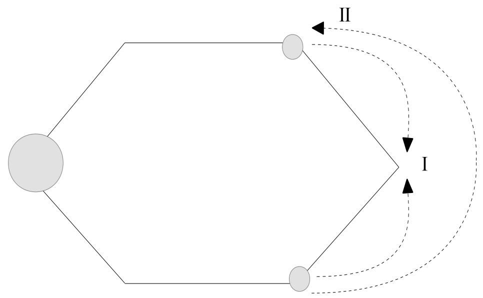

Figure 2: Property 1

Figure 2 depicts the two situations *Property 1* apply to for three alternatives. The two small masses on the right can be joined in a middle way further away than the larger mass (as in the move 1 in the figure), or they can be imbricated at one of the positions with small densities (as in the move 2 in the figure). These types of moves do not decrease (and in case of the move 2, do not increase) the average distance, hence alienation, but increase within group support.

Definition 2 (Property 2). *A polarization measure satisfies* Property 2 *if moving a mass that is opposed mildly towards only smaller masses increases polarization.*

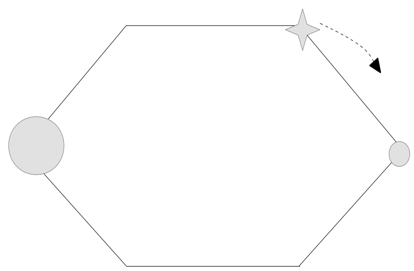

Figure 3: Property 2

Property 2 basically requires that polarization increases whenever an in between stance is moved towards the side which has a smaller support. This idea is explained in Figure 3 for three alternatives. What is carefully required is that the move should be towards only smaller poles and is not carried by a strongly opposed (this strength depends on the whole profile) mass.

Definition 3 (Property 3). *A polarization measure satisfies* Property 3 *if for any profile of preferences with three consecutive masses, dissolution of the middle mass equally into two sides increases polarization.*

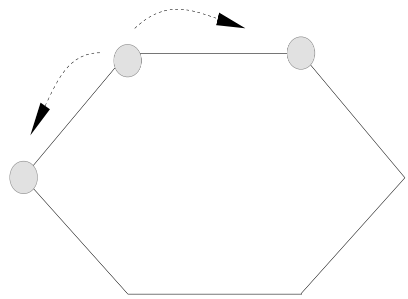

Figure 4: Property 3

Property 3 is an exact counterpart of Axiom 3 in Esteban and Ray [6] and the intuition is clear. The following homotheticity property requires that the size of the society does not matter. Let λP_N denote a profile where $N_i^{\lambda\bar{P}} = \lambda N_i^{\bar{P}}$ for all $i \leq m!$.

Definition 4 (Homotheticity). *Let P_N and $P'_{\tilde{N}}$ be two profiles with possibly distinct sizes; N and $\tilde{N}$. A polarization measure $\mathcal{P}$ satisfies* homotheticity *if $\mathcal{P}(P_N) \geq \mathcal{P}(P'_{\tilde{N}})$ implies $\mathcal{P}(\lambda P_N) \geq \mathcal{P}(\lambda P'_{\tilde{N}})$ for all $\lambda > 0$.*

Finally, *bipolarity*, as a regularity property, requires that full concentration at one order is the least polarized case and that bipolar society where one half is at the exact opposite of the other is the most polarized.

Definition 5 (Bipolarity). *Suppose P_N is such that $\pi_Q = \frac{1}{2} = \pi_{-Q}$ for some $Q \in \mathcal{L}(A)$ and that P'_N is such that $\pi'_{Q'} = 1$ for some $Q' \in \mathcal{L}(A)$. A polarization measure $\mathcal{P}$ is said to satisfy* bipolarity *if $\mathcal{P}(P_N) \geq \mathcal{P}(\dot{P}_N) \geq \mathcal{P}(P'_N)$ for all $N \in \mathbb{N}$ and $\dot{P}_N \in \mathcal{L}(A)^N$.*

2. Results

Proposition 2.1. *The polarization measure defined as*

$$\mathcal{P}^*_\alpha(\pi, \bar{P}) = \sum_{j=1}^{m!} \sum_{i=1}^{m!} \pi_i^{1+\alpha} \pi_j (d_{\mathcal{K}}(P^i, P^j))$$

satisfies properties 1-3 for any $\alpha \in (0, \alpha^*)$ with $\alpha \simeq 1.6$ and $m = 3$.

Note that this function reduces to Gini Index with Kemeny distance when $\alpha = 0$ and the higher α is the higher the sensitivity to identification. In what follows, let's fix $\bar{P} = ((abc) = P^1, (bac) = P^2, (acb) = P^3, (bca) = P^4, (cab) = P^5, (cba) = P^6)$ and denote the initial profile by P_N and the profile after a described move by P'_N.

Proof. *Property 1:* Without loss of generality (wlog), three cases may apply.
(i) $\pi_1 > \pi_4 = \pi_6$, and move $\pi_4 \to P^6$: We have $\mathcal{P}(P'_N) - \mathcal{P}(P_N) = \pi_4(\pi_1^{1+\alpha} - \pi_4^{1+\alpha}) + \pi_4^{1+\alpha}(6\pi_1(2^\alpha - 1))$.
(ii) $\pi_1 > \pi_2 = \pi_4$, and move $\pi_2 \to P^6$: We have $\mathcal{P}(P'_N) - \mathcal{P}(P_N) = \pi_4(\pi_1^{1+\alpha} - \pi_4^{1+\alpha}) + \pi_4^{1+\alpha}(4\pi_1(2^\alpha - 1))$.
(iii) $\pi_1 > \pi_4 = \pi_5$, and move (iii.a) $\pi_4 \to P^5$ or (iii.b) $\{\pi_4, \pi_5\} \to P^6$: (iii.a) induces that $\mathcal{P}(P'_N) - \mathcal{P}(P_N) = 4\pi_1\pi_4^{1+\alpha}(2^\alpha - 1)$ whereas (iii.b) induces $\mathcal{P}(P'_N) - \mathcal{P}(P_N) = 2\pi_4\pi_1^{1+\alpha} + 2\pi_1\pi_4^{1+\alpha}(3 \cdot 2^\alpha - 2)$. In all cases, we have $\mathcal{P}(P'_N) > \mathcal{P}(P_N)$ if $\alpha > 0$.

Property 2: Let π_4 move to P^6 to induce P'_N. We have $\mathcal{P}(P'_N) - \mathcal{P}(P_N) > \pi_4(\pi_1^{1+\alpha} + \pi_2^{1+\alpha} - \pi_3^{1+\alpha} - \pi_5^{1+\alpha} - \pi_6^{1+\alpha}) + \pi_4^{1+\alpha}(\pi_1 + \pi_2 - \pi_3 - \pi_5 - \pi_6)$ which is positive if the move is only towards smaller poles, or formally both $\pi_1 > \pi_6$ and $\pi_2 > \pi_5$, for any support π_3 of opposition to P^4 smaller than ϵ^* where $\epsilon^* = min\{\pi_1 + \pi_2 - \pi_5 - \pi_6, (\pi_1^{1+\alpha} + \pi_2^{1+\alpha} - \pi_5^{1+\alpha} - \pi_6^{1+\alpha})^{\frac{1}{1+\alpha}}\}$ whenever $\alpha > 0$.
Property 3: The question reduces to that of the Axiom 3 in Esteban and Ray [6], and the proof that α is bounded above approximately by 1.6, which pertains to that $max_{z\geq 0}[(1+\alpha)(z - \frac{z^\alpha}{2} - z^{1+\alpha}) - \frac{1}{2}] < 0$, can be found in the last paragraph of the proof of the Theorem 1 in that paper, p. 837. ■

Furthermore, it is quick to observe that the measure $\mathcal{P}^*$ is *anonymous* and *neutral* in the sense that it treats individuals and preferences equally. A polarization measure $\mathcal{P}$ is said to be anonymous if for any profile P_N, for any permutation $\sigma : N \to N$ of individuals, we have that $\mathcal{P}(P_N) = \mathcal{P}(P_{\sigma N})$, where $P_{\sigma N} = (P_{\sigma(i)})_{i\in N}$. Similarly, a polarization measure $\mathcal{P}$ is said to be neutral if for any profile P_N, for any permutation $\delta : A \to A$ of alternatives, we have that $\mathcal{P}(P_N) = \mathcal{P}(\delta P_N)$ where $\delta P_N = (\delta P_i)_{i\in N}$ with $aP_ib \iff \delta(a)\delta P_i\delta(b)$.

Proposition 2.2. *The polarization measure $\mathcal{P}^*_\alpha$ is anonymous, neutral and homothetic for any $\alpha \geq 0$ and $m \in \mathbb{N}$.*

Proof. Take any $N \in \mathbb{N}$ and let $\sigma : N \to N$ be an arbitrary permutation. Since $N_i^P = N_i^{\sigma P}$, we have $\pi_i = \pi'_i$ for all $i \in \mathbb{N}_{m!}$ where $\pi'_i = N_i^{\sigma P}/N$, which demonstrates anonymity. For neutrality it suffices to observe that $d_\kappa(P, P') = d_\kappa(\delta P, \delta P')$ for any $P, P' \in \mathcal{L}(A)$, under any permutation $\delta : A \to A$.[4] The homotheticity follows the fact that the summation is taken over the supports as percentages. ■

Finally, if we take α as 1, which reflects a simple antagonism function where alienation is represented as the distance and identification as just the relative societal support of one's preference[5], the measure satisfies the *bipolarity* condition.

Proposition 2.3. *The measure $\mathcal{P}^*_1$ satisfies bipolarity, for $m = 3$.*

Proof. The measure is zero if we have full concentration at one order, and strictly positive in all other cases. For any two mass profile, increasing distance in between increases polarization clearly. Once they are at exact opposites, making them equal in density will increase the measure.

Lemma 2.1. *The bipolar case is more polarized than any three mass profile under $\mathcal{P}^*_1$.*

Proof. Suppose first that the three masses are not equidistant to each other. Two cases are possible. (*Case i) One is further away from the two.* Let, w.l.o.g., the support be $\{\pi_1, \pi_2, \pi_5\}$. If $\pi_5 > \pi_2$, moving π_1 to P^2 increases the value if $\pi_5^2\pi_1 + \pi_1^2(\pi_5 - \pi_2) + 6\pi_1\pi_2\pi_5 > \pi_2^2\pi_1$ which is true. If $\pi_2 > \pi_5$, moving π_1 to P^5 increases the value if $\pi_2^2\pi_1 + \pi_1^2(\pi_2 - \pi_5) + 3\pi_1\pi_2\pi_5 > \pi_5^2\pi_1$ which is true. (*Case ii) One is at unit distance to both.* Let, w.l.o.g., the support be $\{\pi_1, \pi_2, \pi_3\}$ and that $\pi_2 > \pi_3$. Moving π_1 to P^3 increases the value if $\pi_2^2\pi_1 + \pi_1^2(\pi_2 - \pi_3) + 4\pi_1\pi_2\pi_3 > \pi_3^2\pi_1$ which is true.
Now suppose the three masses are equidistant to each other. Let, wlog, the support be $\{\pi_1, \pi_4, \pi_5\}$ and we move π_1 to P^5. For this to increase polarization it is enough to have π_4 to be the greatest of the three. If

[4] A characterization of Kemeny distance incorporating neutrality can be found in Bogart [3].
[5] Formally, $T(I, a) = a \cdot I$ where $I(\pi_i) = \pi_i$ and $a(d(P^i, P^j)) = d_{\mathcal{K}}(P^i, P^j)$.

they are all equal size $\frac{1}{3}$, we have $\mathcal{P}_1^*(P_N) = 3(\frac{1}{3})^2(2\frac{1}{3} + 2\frac{1}{3}) = \frac{4}{9} < 2(\frac{1}{2})^2(3\frac{1}{2}) = \frac{3}{4}$ where the latter is of the bipolar case. ■

The following lemma shows that from any five or six mass profile, we can reduce the domain to four masses and increase polarization.

Lemma 2.2. *For any six (five) support profile, there exists a move that reduces the support of the profile to five (four) and increases the value of P_1^*.*

Proof. Let masses at each (or all but one) location be positive. Wlog, either (i) $\pi_1 > \pi_4 + \pi_5 + \pi_6$ or (ii) $\pi_5 > \pi_4 + \pi_2 + \pi_1$ is true. Suppose the first is true and move the mass π_2 to P^4.[6] The difference $\mathcal{P}(P_N') - \mathcal{P}(P_N) = \pi_2[\pi_1^2 + \pi_3^2 - (\pi_4^2 + \pi_5^2 + \pi_6^2)] + \pi_2^2[\pi_1 + \pi_3 - (\pi_4 + \pi_5 + \pi_6)]$ is clearly positive. This is regardless of π_3 being non-zero or not (in other words, the support being five or six masses) hence the lemma demonstrated. ■

Corollary 2.1. *For any six support profile there exists a consecutive pair of moves that induces a four support profile with higher value under $\mathcal{P}_1^*$.*[7]

The final lemma below concludes the proof of the proposition by showing that from any (out of three possible) four mass profile we can reduce to a two mass profile with higher measure.

Lemma 2.3. *The bipolar profile is more polarized than any P_N with four masses under $\mathcal{P}_1^*$.*

Proof. Let $\Sigma \subseteq \mathcal{L}(A)$ be the orders that have nonzero support in P_N. Wlog, we have three cases.

(a) $\Sigma = \{P^1, P^3, P^4, P^6\}$. We have three distinct cases; *either (i)* $\pi_4 \geq \pi_3$ and $\pi_1 \geq \pi_6$, *or (ii)* $\pi_4 < \pi_3$ and $\pi_1 < \pi_6$, *or (iii)* $\pi_4 > \pi_3$ and $\pi_1 < \pi_6$ (not distinctively $\pi_4 < \pi_3$ and $\pi_1 > \pi_6$ is also possible.). For the first two cases, moving π_4 to P^6 and π_3 to P^1 increases polarization if $(\pi_4 - \pi_3)(\pi_1 - \pi_6) \geq 0$, which is true. For the third case, moving π_1 to P^4 and π_6 to P^3 increases polarization if $(\pi_4 - \pi_3)(\pi_6 - \pi_1) \geq 0$, which is true.

(b) $\Sigma = \{P^1, P^4, P^5, P^6\}$. Moving π_6 to P^1 induces $\mathcal{P}(P_N') - \mathcal{P}(P_N) > 3\pi_1\pi_6(\pi_1 - \pi_6) > 0$ if $\pi_1 \geq \pi_6$. If otherwise, moving π_4 to P^2 and π_5 to P^3 induces $\mathcal{P}(P_N') - \mathcal{P}(P_N) = (\pi_4 + \pi_5)(\pi_6^2 - \pi_5^2) + (\pi_4^2 + \pi_5^2)(\pi_6 - \pi_1) > 0$ which leaves us with the exact situation as before and hence moving π_1 to P^6 increases polarization.

(c) $\Sigma = \{P^1, P^2, P^4, P^6\}$. Moving π_2 to P^3 induces $\mathcal{P}(P_N') - \mathcal{P}(P_N) = 2\pi_2\pi_4(\pi_2 + \pi_4)$ which is positive and leaves us with the case (a). ■■

3. References

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[6] If the second is true, moving π_6 to P^4 will do the same.

[7] Proof left to the reader, since it only incorporates application of the move in the Lemma 2.4 consecutively.