ECOLE POLYTECHNIQUE

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

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Cahier 2011-16

(Version révisée du cahier 2010-21)

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Overstating: A tale of two cities*

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August 2011

Abstract

If voters vote strategically, is it useful to offer them the possibility of expressing nuanced opinions? Say that a ballot is overstating if it is neither abstention-like nor can be expressed as a mixture of the available ballots. The paper shows that when two additive voting rules share the same (up to an affine transformation) set of overstating ballots, they are strategically equivalent in large elections. It also characterizes "robust" rules, whose set of voting equilibria remains unaltered by adding any finite number of ballots: a rule is robust if and only if it is strategically equivalent to Approval Voting. These results do not hold for small electorates.

KEYWORDS: Strategic voting, voting equilibria.

JEL Classification Numbers: D70; D72.

*This work has benefited from useful discussions with Francis Bloch, Francesco De Sinopoli, Santiago Oliveros and Eduardo Perez-Richet. This work has also benefited from comments of conference and seminar participants in the American Political Science Association Meetting 2011-Seattle, OECD - New Directions in Welfare Conference 2011, the Workshop Advances on Approval Voting 2010, Université Paris II, THEMA - Université de Cergy-Pontoise and Ecole Polytechnique.

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1 Introduction

Shouting and voting are believed to be activities of a different sort. Whereas the latter often goes hand in hand with the idea of finding an agreement (democracy and peace), the former is recurrently associated with the lack of agreement (conflict and wars). Indeed, to put an end to a dispute, one might suggest to switch from shouting to voting. However, even if such a distinction seems to be common sense, both activities have been remarkably close to each other in the past. For instance, Spartans shouted to elect senators to the Gerousia, Sparta's Council of Elders. Each "voter" was allowed to shout as much as he wanted for each of the different candidates. The candidate who had "the most and loudest" acclamations was elected senator, see Girard (2010) [11] for an account of the Spartan voting method.

The Spartan shout is close to Evaluative Voting (EV). Under EV, the voter evaluates each candidate independently on the same numerical scale; the grades are added and the candidate with the largest total is elected. Baujard and Igersheim (2010) [1] report on an experimental work on EV with the scale $\{0, 1, 2\}$. This rule, also called Range Voting, is obviously related to Utilitarianism; see Karni (1998) [13], Dhillon and Mertens (1999) [8], Segal (2000) [27], d'Aspremont and Gevers (2002) [6] and Gaertner and Xu (2010) [9] for axiomatic analysis. A related procedure was in use for several centuries by Venetian oligarchs to elect their Dogi. Instituted in 1268 and used until 1789, the Venetian system allowed voters to express their opinion about each candidate. A voter was given three balls to indicate approval, disapproval or *dubbio*[1]. Another variation of the Spartan method is simply the modern voting rule known as Approval Voting (AV), often advocated for its flexibility since it emerged in the literature in the mid 70s. Approval Voting[2] is Evaluative Voting with the scale $\{0, 1\}$.

Is shouting so different than voting? The previous question can be rephrased in formal voting theory by asking whether extending the set of ballots available under a given voting rule modifies the set of voting equilibria? A voting rule V' is an extension of the voting rule V if all the ballots available under V are available under V'. For instance, Evaluative Voting (the Spartan shout) is an extension of AV. Assuming strategic voting, one might suspect that the set of voting equilibria

[1]The italian *dubbio* corresponds to the English doubt. According to Lines (1986) [17], this doubt is roughly equivalent to an abstention as "a doubt vote, if it ever did exist in doge elections, would essentially be a no vote".

[2]There exists a whole literature on AV. See Laslier and Sanver (2010) [16].

should not be too altered by such an extension. However, this need not be the case for any extension: a well-known extension is known to modify the set of equilibria in a convenient manner. Indeed, AV is an extension of Plurality voting (henceforth PV[3]) and, as shown by Myerson and Weber (1993) [21], AV improves the aggregation of preferences when compared with PV in the noteworthy divided majority situation.

We focus on *additive* voting rules, in which a ballot is a list of points that the voter is affording to the candidates, and where points for each candidate are simply added. (A formal definition of this family of voting rules is provided in the next section.) We analyze such an issue in the context of strategic voting, that is assuming that voters strategically cast their votes in order to maximize their (expected) utility (abstention is allowed). We study equilibria and consider that two voting rules are strategically equivalent if they have the same equilibrium outcomes.

To study small electorates, we use a standard refinement of Nash equilibrium and provide examples that show that voters need not overstate at equilibrium.

To tackle the problem on large elections, we focus on the first and simplest model in this direction, proposed by Myerson and Weber (1993) [21]. In such a model, for any pair of candidates, the voter considers that there is a positive probability that her vote is pivotal on this pair, but some of these probabilities are vanishingly small compared to others. We first define the notion of *strategically equivalent* voting rules. Two rules are strategically equivalent whenever there exists an equilibrium under both of them under which the candidates get the same expected scores (up to a linear transformation) and the pivot probabilities satisfy the same ordering.

We then derive a sufficient condition for the strategic equivalence of voting rules. The condition is simple. Say that a ballot is "overstating" if it is not abstention-like (the ballot does not treat all candidates alike) and if it cannot be expressed as a mixture of other available ballots. If two voting rules offer the same set of overstating ballots, up to an affine transformation, then they are strategically equivalent.

The use of this sufficient condition is fairly straightforward, implying several interesting consequences.

The first consequence is that, in our terms, shouting is voting. The rules used in two cities lead to the same set of voting equilibria: in other words, Approval Voting and Evaluative Voting (EV) are strategically equivalent. The second consequence

[3]In an election held under PV, a voter is allowed to give at most one point to at most one candidate. The candidate with the most votes wins the election. The most used rule for presidential elections is Plurality with a Runoff (Blais (1997) [2]), but we here restrict attention to one-round voting systems.

concerns a different family of voting rules in which no restriction is given over how the grades should be allocated between the different candidates. In an election held under Cumulative Voting (CV), a natural extension of Plurality Voting, a voter is endowed with a finite number of points, and he is allowed to distribute them freely between the different candidates. Different authors[4] have discussed such a method as it gives a high degree of flexibility to the voter. With such a voting rule, voters have the possibility of overstating their vote: that is to give the highest possible amount of points to only one of the candidates. We prove that this is indeed the case in equilibrium, implying that PV and CV are strategically equivalent. We hence prove that for both PV and AV, there exist extensions that do not modify the set of voting equilibria. But, up to now, we have left unaddressed the question of whether there exists voting rules which set of voting equilibria remains unaltered by any finite extension; a voting rule satisfying such a definition is robust[5]. The answer is positive and surprising. Without loss of generality, we work with normalized rules in which the maximum score of a candidate in a ballot is 1. We prove that a voting rule is robust if and only if it is strategically equivalent to AV. To do so, we first show that AV (or any voting rule that contains all the AV ballots) is robust. The reason is simple: any normalized ballot can be expressed as a strict convex combination of AV ballots, and hence our sufficient condition for strategic equivalence applies. Furthermore, any other normalized voting rule which is extended by adding the ballots of AV is strategically equivalent to AV. Hence, any robust voting rule must contain all the AV ballots and hence it is strategically equivalent to AV.

The described equivalence between voting equilibria described is valid along the lines of the theory of large elections proposed by Myerson and Weber (1993) [21]. Nevertheless, small elections (that is elections with few voters) raise new questions. For instance, the information available to voters might be much more detailed in a small election than in a mass election, implying that such theory is of scant interest in the former case. In order to investigate whether the previous claims still hold in environments with few voters, we discuss two voting situations in the case of Evaluative voting. The first example is a pure strategy equilibrium in which there exists a sequence of trembles à la Selten (1975) [28] which induce a voter's unique best response not to be overstating for any arbitrarily small (even though

[4]See Sawyer and McRae (1962) [26], Brams (1975) [4], Nitzan (1985) [22], Cox (1990) [5] and Gerber et. al (1998) [10].

[5]Robustness requires that the set of electoral outcomes is not modified by allowing voters to choose from a wider set of ballots.

positive) perturbation. In the second one, the strategy combination is a mixed-strategy equilibrium in which *the unique pure strategy best response for a voter is not overstating*. Indeed, one of the voters of the election mixes between his undominated strategies making uncertain the final electoral outcome for the rest of the voters. The "mixing" removes the weak preference for overstatements. This situation is stronger that the first one. The situations studied prove that the lack of overstatement can be a best response even in equilibria that satisfy different equilibria refinements. The refinement (trembling-hand perfection) used in this work is among the most classical ways of obtaining equilibria as a limit of games with uncertainty (i.e. perturbed games). However, it is not too difficult to generalize the results to settings in which the uncertainty comes from other sources. For instance, Bayesian games with some uncertainty about voters' types or common values' settings with imperfect information about the true state of nature are good candidates for models in which overstating is not a best response for a strategic voter. Finally, we discuss how the set of possible winners in equilibrium is altered by an extension of the set of ballots. Even though we cannot fully characterize the set of perfect equilibria (due to the lack of structure of voters' mistakes) we are able to show that the equilibria under which "unappealing" candidates win the election both exist under a voting rule and its extension. In other words, the extension of the voting rule does not seem to refine the set of possible winners of the election.

This paper is organized as follows. Section 2 presents the basics of the model. Section 2 to 5 are devoted to large elections: Section 3 describes the equilibrium concept, Section 4 states the sufficient condition for strategic equivalence, Section 5 presents the strategic equivalence between the above-mentioned voting rules, and Section 6 contains the results on the robustness of a voting rule. Section 7 presents the results concerning the environments with few voters, and Section 8 provides some concluding comments.

2 The setting

There are $\mathcal{N}$ voters in the election. Each voter has a type t that determines his strict preferences over the set of candidates $\mathcal{K} = \{1, 2, \dots, \mathcal{K}\}$. The preferences of a voter with type t (a t-voter) is denoted by $u_t = (u_t(k))_{k \in \mathcal{K}}$, in which $u_t(k)$ denotes the utility a t-voter gets if candidate k wins the election. All types t belong to a finite set of types $\mathcal{T}$. The distribution of types is denoted by $r = (r(t))_{t \in \mathcal{T}}$ with $\sum_t r(t) = 1$: in other words, $r(t)$ represents the share of t-voters.

Within this work, we stick to the comparison of additive rules: a ballot is a vector $b = (b_1, b_2, \dots, b_{\mathcal{K}})$ where b_k is the number of points given to candidate k, to be added to elect the candidate with the largest score. Each voter must choose a ballot b from a finite set of possible ballots denoted by B.

For instance, in an election held under *Plurality Voting* (*PV*) voters can abstain or give one point to at most one candidate. Formally:

$$B_{PV} = \{\text{Any permutation of } (1, 0, \dots, 0)\} \cup \{0, 0, ..., 0)\}.$$

Similarly, an *Approval Voting* (*AV*) ballot consists of a vector that lists whether each candidate has been approved or not: for each $j \in \mathcal{K}$, $b_j \in \{0, 1\}$. Hence:

$$B_{AV} = \{0, 1\}^{\mathcal{K}}.$$

Definition 1. *A voting rule V is an* extension *of the voting rule V' if all ballots in V' are available in V, i.e.*

$$B_{V'} \subset B_V.$$

For instance, Approval Voting is an extension of Plurality Voting as $B_{PV} \subset B_{AV}$.

3 Large Elections

We assume that each voter maximizes his expected utility to determine which ballot in the set B he will cast. In this model, his vote has an impact in his payoff if it changes the winner of the election. Therefore, a voter needs to estimate the probability of these situations: the pivot outcomes. We say that two candidates are tied if their vote totals are equal. Furthermore, let H denote the set of all unordered pairs of candidates. We denote a pair $\{i, j\}$ in H as ij with $ij = ji$.

For each pair of candidates i and j, the ij-pivot probability p_{ij} is the probability of the outcome perceived by the voters that candidates i and j will be tied for first place in the election. A voter perceives that the probability that he will change the winner of the election from candidate i to candidate j by casting ballot b with $b_i \geq b_j$ to be linearly proportional to $b_i - b_j$, and that the constant of proportionality (the ij-pivot probability) is the same for the perceived chance of changing the winner from j to i if $b_j \geq b_i$ [6].

[6]This is roughly equivalent to assume that the probability of candidates i and j being tied for first place is the same as the probability of candidate i being in first place one vote ahead candidate

A vector listing the pivot probabilities for all pairs of candidates is denoted by $p = (p_{ij})_{ij \in H}$. This vector p is assumed to be identical and common knowledge for all voters in the election. A voter with ij-pivot probability p_{ij} anticipates that submitting the ballot b can change the winner of the election from candidate j to candidate i to be $p_{ij} \max\{b_i - b_j, 0\}$.

Let $E_t[b]$ denote the expected utility gain of a t-voter from casting ballot b when p is the common vector of pivot probabilities:

$$E_t[b] = \sum_{ij \in H} (b_i - b_j) \cdot p_{ij} \cdot [u_t(i) - u_t(j)]. \tag{1}$$

The expected utility gain from casting ballot b equals the expected utility of casting ballot b minus the expected utility of abstaining. Focusing on utility gains simplifies notation.

A (voting) strategy is a probability distribution σ over the set B that summarizes the voting behavior of voters of each type. For any ballot b and any type t, $\sigma(b \mid t)$ is the probability that a t-voter casts ballot b. Therefore,

$$\tau(b) = \sum_{t \in \mathcal{T}} r(t)\sigma(b \mid t)$$

is the share of the electorate who cast ballot b. Hence, the (expected) score of candidate k is

$$S(k) = \sum_{b \in B} b_k \tau(b).$$

The set of likely winners of the election contains the candidates whose expected score $S(k)$ is maximal given the strategy σ.

Myerson and Weber (1993) [21] assume that voters expect candidates with lower expected scores to be less likely serious contenders for first place than candidates with higher expected scores. In other words, if the expected score for some candidate l is strictly higher than the expected score for some candidate k, then the voters would perceive that candidate l's being tied with any third candidate m is much more likely than candidate k's being tied for first place with candidate m[7].

j (and both candidates above the rest of the candidates), which is in turn the same one as the probability of candidate j being in first place one vote ahead candidate i. Myerson and Weber (1993) [21] justify this assumption by arguing that it seems reasonable when the electorate is large enough. This is not verified in Poisson games, a formal model of large elections in which the pivot probabilities are derived endogenously from the structure of the game.

[7] This assumption is needed in order to ensure the existence of equilibrium. The results presented here do not lie on the ordering of the pivot probabilities.

Definition 2. *Given a (voting) strategy σ and any $0 < \varepsilon < 1$, a pivot probability vector p satisfies the* ordering condition *for ε (with respect to σ) if, for every three distinct candidates i, j and k:*

$$S(i) > S(j) \Longrightarrow p_{jk} \leq \varepsilon p_{ik}.$$

Besides, Myerson and Weber (1993) [21] assume that the probability of three (or more) candidates being tied for first place is infinitesimal in comparison to the probability of a two-candidate tie.

Definition 3. *The strategy σ is a (voting)* equilibrium *of the game if and only if, for every positive number ε, there exists some vector p of positive pivot probabilities that satisfies the ordering condition and such that, for each ballot b and for each type t,*

$$\sigma(b \mid t) > 0 \implies b \in \arg\max_{d \in B} E_t[d].$$

It should be stressed that, in this definition, the pivot probabilities p_{ij} are supposed to be constant when the voter contemplates casting one ballot or the other. This point will play an important role in the next section. It is justified when the number of voters is large for, in that case, the voter cannot change with his single vote the order of magnitude of these probabilities. It can be shown that the set of equilibria is non-empty[8].

Finally, an important concept in our model should be defined: the equivalence between equilibria under different voting rules.

Definition 4. *An equilibrium σ_U of an election held under a voting rule U is* equivalent *to an equilibrium σ_V of the same election held under V if and only if*

1. *the pivot probabilities satisfy the same ordering and*

2. *the scores of the candidates coincide, up to an affine transformation.*

The sets of voting equilibria of an election held under two voting rules U and V are equivalent if for any voting equilibrium of the election held under U (resp. V), there exists an equivalent voting equilibrium of the election held under V (resp. U)

Definition 5. *Two voting rules are* strategically equivalent *if and only if their set of voting equilibria are equivalent.*

[8]See Theorem 1, page 105 in Myerson and Weber (1993) [21].

We will pay special attention to the set of possible winners W_V that arise under a voting rule V. A possible winner is a candidate who wins the election in equilibrium with positive probability. The set of *possible winners* of an election held under the voting rule V is

$$W_V = \{k \in \mathcal{K} \mid \text{There exists an equilibrium } \sigma \text{ in which } S(k) \text{ is maximal}\}.$$

If two voting rules are strategically equivalent, then they have the same set of possible winners. However, the converse need not be true; for instance, the rankings of other candidates may differ. It is noteworthy that the definition of strategic equivalence used is rather demanding. It requires more than the set of possible winners being the same under two voting rules. This demanding definition reinforces our results as we show that this strong version of equivalence holds in the Myerson-Weber setting.

4 A sufficient condition for strategic equivalence

We now introduce some categorizations of the ballots that will be useful throughout.

An *abstention* ballot is a ballot with all the coordinates alike ; the set of such ballots is denoted by $\text{Abs}(B)$.

An *interior* ballot b is a ballot which is not an abstention ballot and that can be expressed as a strict convex combination of other ballots in B, i.e. there exist ballots

$$b^1, b^2, \ldots, b^m \in B \text{ with } b = \sum_i \alpha_i b^i \text{ with } \alpha_i \in (0,1) \text{ and} \sum_i \alpha_i = 1.$$

An *overstating* ballot is a ballot which is neither an interior nor an abstention ballot. Given the set of ballots B, the set of interior and overstating ballots are respectively denoted by $\text{Int}(B)$ and $\text{Ove}(B)$ with,

$$\text{Ove}(B) = B \setminus \{\text{Int}(B) \cup \text{Abs}(B)\}.$$

Remark 1.

Casting an abstention ballot or a ballot which is a convex combination of at least one abstention ballot is dominated for every voter.

To see this, let b^j denote an abstention ballot with $b^j = (x, \ldots, x)$. By formula (1), $E[b^j] = \sum_{ij \in H} (x - x) \cdot p_{ij} \cdot [u_t(i) - u_t(j)] = 0$ so that any abstention ballot is dominated. Let $c = \sum_i \alpha_i b^i$ denote the convex combination of at least one abstention ballot so that $E_t[c] = \sum_{i \neq j} \alpha_i E_t[b^i]$ as $E_t[b^j] = 0$. If $E_t[c] \leq 0$, then casting ballot c is dominated. If $E_t[c] > 0$, then $\sum_{i \neq j} \alpha_i E_t[b^i] > 0$ so that there must exist b^i with $i \neq j$ with $E_t[b^i] > 0$. Then, casting ballot c is dominated by the mixed strategy that casts ballot c with probability $1 - \alpha_j$ and ballot b^i with probability α^j.

In other words, neither an abstention ballot nor a ballot which is a convex combination of at least one abstention ballot are cast with positive probability in equilibrium.

Remark 2.

The definition of interior ballot implies that if a ballot c is interior then it is the strict convex combination of overstating and abstention ballots:

$$c = \sum_{b \in B'} \alpha_b \cdot b \text{ for some } B' \text{with } B' \subset \text{Ove}(B) \cup \text{Abs}(B) \text{ with } \alpha_b \in (0,1) \text{ and } \sum_{b \in B'} \alpha_b = 1.$$

Remark 3.

The ballot set B is *finite* and hence not every ballot can be expressed by a convex combination of other ballots in B.

Remark 4.

The set $\text{Ove}(B)$ of overstating ballots is non-empty for any non-trivial voting rule. To see this, let us suppose that $\text{Ove}(B) = \emptyset$ for some voting rule with ballot set B. By definition, $B = \text{Ove}(B) \cup \text{Int}(B) \cup \text{Abs}(B)$. By Remark 3, not every ballot can be expressed by a convex combination of other ballots in B, so that $B \neq \text{Int}(B)$. As we have assumed that $\text{Ove}(B) = \emptyset$, we must have that $\text{Abs}(B) \neq \emptyset$, i.e. every ballot which is not interior is an abstention ballot. Hence, every ballot of such a rule is an abstention ballot as any interior ballot is a strict convex combination of other ballots. Thus, such a voting rule can be labeled as trivial as it elects for any preference profile the whole set of candidates.

Example 1.

Let us consider a three-candidates election held under Cumulative Voting. We assume that a voter is endowed with at most two points that can be freely distributed among the different candidates. The set of allowed ballots B_{CV} is:

$$B_{CV} = \{(0,0,0),(2,0,0),(0,2,0),(0,0,2),\\ (1,1,0),(1,0,1),(0,1,1),(1,0,0),(0,1,0),(0,0,1)\}.$$

The interior ballots are the following ones:

$$\text{Int}(B_{CV}) = \{(1,1,0),(1,0,1),(0,1,1),(1,0,0),(0,1,0),(0,0,1)\},$$

whereas there is a unique abstention ballot:

$$\text{Abs}(B_{CV}) = \{(0,0,0)\}.$$

(To see why, it is enough to write that for instance the interior ballot $(1,1,0)$ equals the convex combination $1/2(2,0,0) + 1/2(0,2,0)$.) Finally, the set of overstating ballots is:

$$\text{Ove}(B_{CV}) = \{(2,0,0),(0,2,0),(0,0,2)\}.$$

4.1 Overstating

By Remark 1, an abstention ballot is not cast with positive probability in equilibrium so that there are two types of equilibria: interior and overstating. We refer to an *interior equilibrium* whenever a voter's best response includes an interior ballot. On the contrary, an *overstating equilibrium* is an equilibrium in which every voter's best response uniquely includes overstating ballots.

Proposition 1. *[Overstating] For any interior equilibrium, there exists an equivalent overstating equilibrium.*

Proof. Let σ be an interior equilibrium such that some t-voter's best response satisfies $\sigma(c \mid t) = 1$ with c an interior ballot. Therefore we can write $c = \sum_{b \in B'} \alpha_b \cdot b$ for some subset B' of the set of overstating ballots as c is cast with positive probability at equilibrium (Remarks 1 and 2).

Formula (1) implies that

$$E_t[c] = E_t[\sum_{b \in B'} \alpha_b \cdot b] = \sum_{b \in B'} \alpha_b \cdot E_t[b].$$

In other words, the t-voter is indifferent between casting ballot c and playing a mixed strategy over the set B' which mimics the convex combination that defines ballot c. Formally, the voter t is indifferent between strategy σ as defined[9] and strategy σ^* with

$$\begin{cases} \sigma^*(b \mid t) = \alpha_b & \text{for all } b \in B' \text{ and} \\ \sigma^*(\cdot \mid t') = \sigma(\cdot \mid t') & \text{for voters with type } t' \neq t. \end{cases}$$

Besides, for every $\varepsilon > 0$, the pivot probability vector p that justifies the strategy σ also justifies the strategy σ^* as the scores of candidates coincide under both strategies, implying that σ^* is an equilibrium.

All in all, both σ and σ^* are justified by the same pivot probability vector and under both of them, the expected scores of the candidates coincide. Hence, for any interior equilibrium σ, there exists an equivalent overstating equilibrium σ^*. □

4.2 Strategic equivalence

Proposition 1 proves that interior equilibria are not informative in the sense that they do not add any information regarding the equilibria that can be attained under a voting rule. Building on such a result, we now give a simple sufficient condition to ensure the strategic equivalence of two voting rules. Indeed, we show that if there exists a bijective affine transformation between the overstating ballots of two voting rules, then both rules are strategically equivalent.

Theorem 1. *Whenever there exists a bijective affine transformation between the overstating ballots $Ove(B_U)$ and $Ove(B_V)$ of two voting rules U and V, U and V are strategically equivalent.*

Proof. Let U and V denote two voting rules such that there exists a bijective affine transformation $f : b \longmapsto f(b) = \alpha + \beta \cdot b$, from $\text{Ove}(B_U)$ onto $\text{Ove}(B_V)$, for some reals α and β.

Let σ be a strategy in an election held under U in which every voter only casts ballots in the set $\text{Ove}(B_U)$. Let σ^* denote a strategy in the same election held under V that satisfies

$$\sigma^*(b^* \mid t) = \sigma(b \mid t), \tag{2}$$

in which each ballot b^* satisfies $b^* = f(b) = \alpha + \beta b$.

[9] Throughout the proof, t-voters play in pure strategies in the equilibrium σ. Similar arguments can be used to extend the proof whenever σ involves that some t-voters play in mixed strategies.

Let us now prove that the strategy σ is an overstating equilibrium of the election held under U if and only if the strategy σ^* is an overstating equilibrium of the election held under V with both σ and σ^* being equivalent.

First of all, an equilibrium is overstating if and only if the unique ballots that are cast with positive probability are overstating. Thus, if we show that σ and σ^* are equilibria under U and V respectively, then both equilibria will be overstating.

Let us now prove that casting ballot b is a best response given σ if and only if casting ballot b^* is a best response given σ^*:

$$\begin{aligned} b \in \arg \max_{d \in \text{ Ove}(B_U)} E_t[d] &\iff E_t[b] \geq E_t[d] \ \forall\, d \in \text{ Ove}(B_U) \\ &\iff \alpha + \beta E_t[b] \geq \alpha + \beta E_t[d] \ \forall\, d \in \text{Ove}(B_U) \\ &\iff E_t[b^*] \geq E_t[d^*] \ \forall\, d^* \in \text{Ove}(B_V) \\ &\iff b^* \in \arg \max_{d^* \in \text{ Ove}(B_V)} E_t[d^*]. \end{aligned}$$

Besides, given that the strategy σ^* satisfies (2), the scores of the candidates $S(\cdot)$ given σ and $S^*(\cdot)$ given σ^* satisfy

$$S^*(k) = \alpha + \beta S(k) \ \forall\, k \in \mathcal{K},$$

and whence the scores of candidates coincide up to an affine transformation under both strategies.

In order to prove the equivalence between σ and σ^*, it remains to be checked that pivot probabilities satisfy the same ordering with both strategies σ and σ^*. However, as the scores of candidates coincide up to an affine transformation with both strategies, a pivot probabilities vector p satisfies the ordering condition under σ if and only if it satisfies the ordering condition under σ^*.

We have proved so far that σ is an overstating equilibrium under U if and only if there exists an equivalent overstating equilibrium σ^* under V. In other words, if there exists a bijective affine transformation between $\text{Ove}(B_U)$ and $\text{Ove}(B_V)$, the set of overstating equilibria under both U and V are equivalent. But the previous equivalence finishes the proof as by Proposition 1, any interior equilibrium under a voting rule is equivalent to an overstating equilibrium under the same voting rule.

□

Theorem 1 has the advantage of being extremely simple to use: indeed, as will be shown the next section, almost no computation is needed to check the strategic equivalence of two voting rules.

5 Applications

Two applications of Theorem 1 are now described. The main interest of such a theorem is that it allows to "simplify" voting rules, in which the term simplify has been coined by the recent literature on mechanism simplification[10]. In this literature, a mechanism is simplified by reducing the message space of the agents, while no new equilibria are created as a consequence of this reduction. When the number of voters becomes large enough, adding or removing *interior* ballots to a voting rule does not modify the set of voting equilibria. Our results hence prove that when the number of voters is large enough, many voting rules can be simplified.

5.1 Evaluative Voting: One man, Many extended votes

An AV ballot consists of a vector that lists whether each candidate has been approved or not: $\forall j \in K$, $b_j \in \{0,1\}$. Hence it is simple to see that $\text{Abs}(B_{AV}) = \{(0,\dots,0),(1,\dots,1)\}$.

Under Evaluative Voting, a voter can assign up to m points to each candidate for some positive m. Hence,

$$b \text{ is an } EV \text{ ballot if } \forall\, j \in \mathcal{K},\ b_j \in \{0,1,\dots,m\},$$

with abstention ballots $\text{Abs}(B_{EV}) = \{(0,\dots,0),(m,\dots,m)\}$.

Theorem 2. *EV and AV are strategically equivalent.*

Proof. The set of overstating ballots of AV satisfies

$$\text{Ove}(B_{AV}) = \{0,1\}^{\mathcal{K}} \setminus \{(0,\dots,0),(1,\dots,1)\},$$

and the set of overstating ballots of EV equals

$$\text{Ove}(B_{EV}) = \{0,m\}^{\mathcal{K}} \setminus \{(0,\dots,0),(m,\dots,m)\},$$

so that, by Theorem 1, EV and AV are strategically equivalent. □

5.2 Cumulative Voting: One man, One extended vote

In an election held under PV, voters can give one point to at most one candidate. Formally, we say that b is a PV ballot if for every $j \in \mathcal{K}$, $b_j \in \{0,1\}$ *and* there is at most one $b_j \neq 0$.

[10]See Milgrom (2009, 2010) [18, 19] and Perez-Richet (2011) [25].

In an election held under *Cumulative Voting*, a voter can assign up to m points to each candidate for some positive m with the restriction that the sum of the points he can assign to each of the candidates is at most m. Hence,

$$B_{CV} = \left\{ b_j \in \{0, 1, \dots, m\} \, \forall \, j \in \mathcal{K} \text{ and } \sum_{k \in \mathcal{K}} b_k \leq m \right\}.$$

Theorem 3. *CV and PV are strategically equivalent.*

Proof. Given the set of ballots under PV and CV, one obtains that

$$\text{Ove}(B_{PV}) = \{\text{Any permutation of } (1, 0, \dots, 0)\}.$$

and that

$$\text{Ove}(B_{CV}) = \{\text{Any permutation of } (m, 0, \dots, 0)\}.$$

which, by Theorem 1, concludes the proof. □

6 Robust Voting rules

Within this section, we introduce the notion of robust voting rule and show that a voting rule is robust if and only if it is strategically equivalent to AV.

Definition 6. *A voting rule V is robust if any extension of V is strategically equivalent to V.*

Robustness requires that the set of electoral outcomes is not modified by allowing voters to choose from a wider set of ballots.

A voting rule is normalized if the maximum score of a candidate in a ballot is 1. Remark that any ballot of a normalized rule can be expressed as the mixture (convex combination) of AV ballots. Hence, for any normalized voting rule V which extends AV, we have

$$\text{Ove}(B_{AV}) = \text{Ove}(B_V),$$

implying, by Theorem 1, the strategic equivalence of V and AV as summarized by the next result. Furthermore, extending any voting rule V by "adding" the AV ballots modifies the set of overstating ballots and hence leads to a voting rule which is strategically equivalent to AV. We hence can state without proof the following characterization of robust voting rules.

Theorem 4. *A voting rule is robust if and only if it is strategically equivalent to AV.*

7 Small Elections

The results previously presented are a consequence of the model used in which voters' perceptions over the impact of their ballots in switching the winner of the election have a very specific shape. Such a theory fits particularly well the study of mass elections. Indeed, as shown by further developments of the theory[11], more formal models give, roughly speaking, similar predictions depending on whether the ordering condition is satisfied. However, it seems that the specific shape of expected utility is not particularly relevant for studying voting in committees (that is voting with few voters). Indeed, in a committee, the information a voter knows can be much more detailed than in a large election.

When switching to an environment with few voters, we prove two results:

- Not overstating might be the unique best response for a strategic voter.
- The set of winners of an election is not significantly altered by allowing voters to express an intensity of preference.

7.1 Overstating need not be optimal

In order to prove that not overstating may be a unique best response in a voting game with few voters, we present two examples. The first one presents a pure strategy equilibrium whereas the second one is a mixed strategy one. In order to test the robustness of the examples, we focus on trembling-hand perfection à la Selten. The definition of perfection is as follows:

Definition 7. *A completely mixed strategy profile $\sigma^{\varepsilon}_{\mathcal{N}}$ is an ε-perfect equilibrium in an $\mathcal{N}$-voters game if*

$$\forall i \in \mathcal{N},\ \forall b^i, \bar{b}^i \in B,\ \textit{if}\, U_i(b^i, \sigma^{\varepsilon}_{\mathcal{N}\setminus\{i\}}) > U_i(\bar{b}^i, \sigma^{\varepsilon}_{\mathcal{N}\setminus\{i\}}),\ \textit{with}\ \sigma^{\varepsilon}(\bar{b}^i) \leq \varepsilon,$$

in which $U_i(b)$ denotes the payoff of voter i given the strategy combination b. We refer to the strategy combination $\sigma_{\mathcal{N}}$ as a perfect equilibrium if there exists a sequence $\{\sigma^{\varepsilon}_{\mathcal{N}}\}$ of ε-perfect equilibria converging (for $\varepsilon \to 0$) to $\sigma_{\mathcal{N}}$.

[11]See Myerson (2002) [20], Laslier (2009) [15], Núñez (2009) [24], Bouton and Castanheira (2010) [3], Goertz and Maniquet (2010) [12], and Nuñez (2010) [23].

Example 1:

There are three candidates $\mathcal{K} = \{1, 2, 3\}$ and three different types $\mathcal{T} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, with cardinal utilities given by:

$$u_{\mathtt{a}} = (3, 1, 0), \quad u_{\mathtt{b}} = (0, 3, 1) \text{ and } u_{\mathtt{c}} = (0, 1, 3).$$

There are seven voters in the electorate. Voters **1** and **2** have type a, voters **3** and **4** have type b and voters **5**, **6** and **7** have type c.

We consider Evaluative Voting in which voters can give up to two points to each of the candidates.

We let f denote the strategy combination with

$$f = ((2, 0, 0), (2, 0, 0), (0, 2, 1), (0, 2, 1), (0, 0, 2), (0, 0, 2), (0, 0, 2)).$$

It is simple to see that voters **3** and **4** do not overstate. Besides, f is an equilibrium in undominated strategies in which candidate 3 wins the election by more than two votes.

Proposition 2. *In Example 1, f is a perfect equilibrium in which some voters' best responses are not overstating.*

This proposition, the proof of which is included in the appendix, shows that not overstating might be a voter's best response in a perfect equilibrium. Indeed, a perfect equilibrium is the limit of completely mixed strategies of the voters that arise as a consequence of uncorrelated mistakes of the voters. Hence, voters' expected utility is not anymore "smooth" as it is by assumption in the large elections model.

Example 2:

There are three candidates $\mathcal{K} = \{1, 2, 3\}$ and four different types $\mathcal{T} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$, with cardinal utilities given by:

$$u_{\mathtt{a}} = (6, 1, 0), \quad u_{\mathtt{b}} = (0, 6, 1), \;\; u_{\mathtt{c}} = (0, 1, 6) \text{ and } u_{\mathtt{d}} = (0, 3, 6).$$

There are seven voters in the electorate. Voters **1**,**2** and **3** have type a, voters **4** and **5** have type b, voter **6** has type c and voter **7** has type d .

We consider Evaluative Voting in which voters can give up to two points to each of the candidates.

We let g denote the strategy combination

$$g = ((2, 0, 0), (2, 0, 0), (2, 0, 0), (0, 2, 1), (0, 2, 1), (0, 0, 2), g_7).$$

in which g_7 stands for the mixed strategy $1/3(0,0,2) + 1/3(0,1,2) + (1/3)(0,2,2)$ of voter **7**. Every voter plays an undominated strategy in the strategy combination g. It is easy to check that g is a mixed-strategy equilibrium of the election in which voters **1** to **6** are playing a unique best response.

Proposition 3. *In Example 2, g is a perfect equilibrium in which some voters' unique best responses are not overstating.*

Remark 1: The source of the non overstating behavior in the second example is clearly the uncertainty faced by voters **4** and **5**, as a consequence of the mixing of voter **7**. The same logic applies in a (Bayesian) game of incomplete information in which voters are not sure of the type of their opponents.

Remark 2: The utility vectors in both examples are consistent with single-peaked preferences.

7.2 Possible Winners remain unchanged

We now address the issue of the set of possible winners in an election with a small number of voters. To do so, we give a proposition which extends a previous result of De Sinopoli (2000) [7] (which focused in Plurality Voting). We show that any candidate who is not a Condorcet loser can win the election under Plurality Voting and Cumulative Voting.

Prior to stating it, we need the definition of Condorcet loser.

Definition 8. *Candidate k' is a Condorcet loser if*

$$\#\{i \in \mathcal{N} \mid u_i(k) > u_i(k')\} > \#\{i \in \mathcal{N} \mid u_i(k') > u_i(k)\} \;\; \forall\, k \;\in\; \mathcal{K} \setminus k'.$$

Proposition 4. *In an election held under either PV or CV with at least 4 voters, for every candidate k who is not a Condorcet loser there exists a perfect equilibrium in which k wins the election.*

Proof. Let 1 and 2 be two candidates who are not Condorcet losers. Let us divide the voters in two groups: the voters who prefer candidate 1 to candidate 2, $V(1,2) = \{i \in \mathcal{N} \mid u_i(1) > u_i(2)\}$, and the remaining ones $V(2,1) = \{i \in \mathcal{N} \mid u_i(2) > u_i(1)\}$. Under CV, a voter can assign up to m points to a single candidate. Under PV, the proof remains unchanged with the constraint that $m = 1$. Consider the mixed strategy d^{ε} such that for every voter $i \in V(1,2)$, where η_i denotes the mixed strategy of voter that assigns equal probability to all his pure strategies with obvious notations,

$$d_i^\varepsilon = (1-\varepsilon-\varepsilon^2)(m,0,\dots,0) + \varepsilon(0,m,0\dots,0) + \varepsilon^2\eta_i,$$

and such that for every voter $i \in V(2,1)$,

$$d_i^\varepsilon = (1-\varepsilon-\varepsilon^2)(0,m,\dots,0) + \varepsilon(m,0,0\dots,0) + \varepsilon^2\eta_i.$$

For each voter, the pivot event which becomes infinitely more likely as ε tends towards zero is one in which candidates 1 and 2 are involved.[1] Hence, each voter plays his best response with probability higher than ε in the sequence of mixed strategies d^ε. Besides, as ε approaches zero, every voter in the set $V(1,2)$ votes for candidate 1, and every other voter votes for candidate 2, which implies that either candidate 1 or candidate 2 wins the election, proving the claim. □

The previous result implies that extending the set of available grades in the case of PV does not refine in a relevant way the set of possible winners of elections with few voters.

The reason why the equilibrium depicted by the Proposition 4 can be constructed is simple. For any pair of candidates 1 and 2 (who are not Condorcet losers), we split the electorate in two blocs: the ones who prefer candidate 1 to candidate 2 (the partisans of candidate 1) and the ones who prefer candidate 2 to candidate 1 (the partisans of candidate 2). Let us assume that partisans of candidate 1 assign her the maximum number of points whereas partisans of candidate 2 behave in the same manner with respect to candidate 2. Each of the two blocs is homogenous in the sense that each voter makes the same mistakes. Hence, when casting his ballot, a voter knows almost surely that, provided being pivotal, his vote will break the close race between candidates 1 and 2. Therefore, it is a best response for the partisans of a candidate to assign her the maximum number of points, proving that this is an equilibrium. Both voting rules analyzed in the Proposition 4 share the feature that a voter can assign the total number of points to a single candidate, leading to the construction of this "almost-everything-can-happen" type of result.

The Majority Preferred Candidate In order to conclude our investigation in the case of a reduced number of voters, we focus on the majority preferred candidate situation, in a similar spirit to the one depicted by Nuñez (2010) [23]. Let us consider a voting game held under Evaluative Voting. There are three types of voters in the electorate:

$$u_{\mathtt{a}} = (3,0,1), u_{\mathtt{b}} = (1,3,0) \text{ and } u_{\mathtt{c}} = (1,0,3),$$

with voters 1,2 being of type `a`, voters 3 to 7 being of type `b` and voters 8 to 10 of the third type. We will refer to candidate 2 as the *majority preferred candidate* as 5 voters over 10 rank him first. Candidate 1 is only ranked as a first option by two over ten voters in the election but it can nevertheless be elected at equilibrium in elections held under EV and AV as shown by next result.

Proposition 5. *There exists a perfect equilibrium in which candidate 1 is the unique winner of the election held under both EV and AV.*

Proof. Under EV, a voter can assign up to m points to a single candidate. Consider the mixed strategy e^{ε} where η_i denotes the mixed strategy of a voter that assigns equal probability to all his pure strategies,

$$e_i^{\varepsilon} = (1-\varepsilon-\varepsilon^2)(m,0,0) + \varepsilon^2\eta_i \text{ with } i = 1,2$$
$$e_i^{\varepsilon} = (1-\varepsilon-\varepsilon^2)(m,m,0) + \varepsilon(0,0,m) + \varepsilon^2\eta_i \text{ with } i = 3,\ldots,7,$$
$$e_i^{\varepsilon} = (1-\varepsilon-\varepsilon^2)(0,0,m) + \varepsilon^2\eta_i \text{ with } i = 8,\ldots,10.$$

For each voter $i = 1,2$, the pivot event which becomes infinitely more likely as ε tends towards zero is $(4m, 3m, 5m)$ so that it is a strict best response to vote only for his first-ranked candidate. Similarly, for each voter $i = 3,\ldots,7$, the pivot event which becomes infinitely more likely as ε tends towards zero is $(4m, 2m, 5m)$ so that it is a strict best response to vote for his first-ranked and his second-ranked candidate. Finally, the event that determines voters $i = 8, 9, 10$'s best responses is $(5m, 3m, 4m)$ and hence their unique best response is to cast ballot $(0, 0, m)$. Besides, as ε approaches zero, candidate 1 wins the election as every voter who votes for candidate 2 also votes for candidate 1, proving the claim. □

The bottom-line of this example is that even if we do not provide a characterization of possible winners under Evaluative Voting, enlarging the set of possible grades does not remove the coordination problems already present under Approval Voting. Hence, one can intuitively think that the set of possible winners should not be too refined by EV (when compared to AV), if at all. Similar coordination problems as the ones illustrated by Proposition 5 have been already identified by Nuñez (2010) [23] in the case of AV. The logic of this unattractive equilibrium boils down to voters' anticipations. In a certain manner, AV performs better than PV in preference aggregation as, with the former voting rule the voter does not face the classical trade-off between voting for his preferred candidate and voting for his preferred likely winner (the wasted-vote effect). However, this property of AV

(and of EV) may not be enough to ensure a correct preference aggregation in every election. If the majority of voters anticipate that their preferred candidate is not included in the most probable pivot outcome, this may lead to the election of an unappealing candidate. Indeed, due to their anticipations, the majority of voters favors their preferred likely winner by assigning her the maximum number of points and at the same time vote for their preferred candidate, leading to the election of the former candidate.

8 Conclusion

Building on the theory of strategic voting in large elections, we have derived a sufficient condition for the strategic equivalence of voting rules that simply depends on the ballots available to the voters. The condition says that whenever two voting rules share the same set of overstating ballots (up to an affine transformation), then they are strategically equivalent. Hence, such a condition helps us to draw some conclusions over how adding ballots to a given voting rule modifies the set of voting equilibria. First, we prove that it is possible to add ballots to both Plurality Voting and Approval Voting without modifying the set of voting equilibria. In the case of Approval voting, there is no difference between shouting (the Spartan Shout) and voting (AV) when voters act strategically. As far as Plurality voting (PV) is concerned, Cumulative voting extends PV while being strategically equivalent to PV. We then characterize robust voting rules, voting rules which set of voting equilibria is not modified by adding any finite number of ballots. We show that any robust voting rule is strategically equivalent to AV. As has been shown, the previous results do not extend to a context with a reduced number of voters.

We have very few observations to back up, or to invalidate, these theoretical results. Laslier and Van der Straeten (2004) [14] report on an experiment comparing EV with the 0 to 10 scale and AV, and Baujard and Igersheim (2010) [1] report on an experiment comparing EV with the 0-1-2 scale and AV. In both cases it is observed that the outcome of the election (the elected candidate) is the same under the two systems, even if it is not observed that voters concentrate on extreme grades.

An interesting extension of the present work would be to understand whether similar results apply under proportional representation or in multi-seat elections in which voters have to distribute their votes. Finally, it is noteworthy to underline that our sufficient condition for strategic equivalence remains silent over the different rank scoring rules as characterized by Young (1975) [30]. Indeed, there are no interior ballots in a rank scoring rule and the different scoring rules do not share the same set of overstating ballots.

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A Appendix: Proof of Proposition 2

In an election held under Evaluative Voting in which voters can give up to two points to each of the three candidates, voters have three undominated strategies: to give two points to their favorite candidate, no points to their least preferred candidate and zero, one or two points to their middle ranked candidate. The previous observation is important, as in a perfect equilibrium voters only choose undominated strategies[12]. It is easy to see that the strategy combination

$$f = ((2,0,0),(2,0,0),(0,2,1),(0,2,1),(0,0,2),(0,0,2),(0,0,2))$$

is an undominated equilibrium in which b-voters do not overstate and in which candidate 3 wins the election. Consider the following completely mixed strategy combination f^ε, where η_i denotes the mixed strategy of voter i which assigns equal probability to all his pure strategies.

$i = \mathbf{1}, \mathbf{2}$ $f_i^\varepsilon = (1-27\varepsilon^2)(2,0,0) + 27\varepsilon^2\eta_i$

$i = \mathbf{3}, \mathbf{4}$ $f_i^\varepsilon = (1-27\varepsilon^2)(0,2,1) + 27\varepsilon^2\eta_i$

$i = \mathbf{5}, \mathbf{6}, \mathbf{7}$ $f_i^\varepsilon = (1-\varepsilon_1-\varepsilon_2-25\varepsilon^2)(0,0,2) + (\varepsilon_1-\varepsilon^2)(2,0,0) + (\varepsilon_2-\varepsilon^2)(2,2,0) + 25\varepsilon^2\eta_i$,

in which $\varepsilon_1 = 1/3(\varepsilon+\varepsilon^2)$ and $\varepsilon_2 = 1/3(2\varepsilon-\varepsilon^2)$.

It is easy to see that, for ε sufficiently close to zero, this is an ε-perfect equilibrium. Suppose all voters other than i choose the strategies prescribed by f. Then,

[12] See Corollary 2.2.6, page 29 in van Damme (1996) [29].

the three undominated strategies of voter i are equivalent. Since for ε going to zero, the probability of voter **5** (the same statement is valid for voters **6** or **7**) to tremble towards $(2,0,2)$ or $(2,2,0)$ is infinitely greater than the probability of any other mistake, due to the trembling of one or several voters, it is enough to check that in both of these events the limiting strategy is preferred to the other undominated strategy.

For voters **1** and **2**, the relevant contingencies which allow them to discriminate between their three undominated strategies is when the behavior of the others is summarized by the vectors $(4,4,6)$ and $(4,6,6)$. Let us denote their probabilities given voter's best responses by $p((4,4,6) \mid f^{\varepsilon}_{-i})$ and $p((4,6,6) \mid f^{\varepsilon}_{-i})$. Furthermore, given voter's best responses, we can write that $2p((4,4,6) \mid f^{\varepsilon}_{-i}) = p((4,6,6) \mid f^{\varepsilon}_{-i})$. Since

$$\begin{aligned} U_{\mathbf{1}}(2,0,0) &= 3/2\, p((4,4,6) \mid f^{\varepsilon}_{-i}) + 4/3\, p((4,6,6) \mid f^{\varepsilon}_{-i}) \\ &= 25/12\, p((4,6,6) \mid f^{\varepsilon}_{-i}) \\ &> U_{\mathbf{1}}(2,1,0), U_{\mathbf{1}}(2,2,0). \end{aligned}$$

Hence, (2,0,0) is the best reply to f^{ε}_{-i}. The same statement is true for voter **2**.

For voters **3** and **4**, the relevant contingencies can be summarized by the vectors $(6,2,5)$ and $(6,4,5)$. Let us denote their probabilities by $p((6,2,5) \mid f^{\varepsilon}_{-i})$ and $p((6,4,5) \mid f^{\varepsilon}_{-i})$. Furthermore, given voter's best responses, we can write that $2p((6,2,5) \mid f^{\varepsilon}_{-i}) = p((6,4,5) \mid f^{\varepsilon}_{-i})$. Since

$$\begin{aligned} U_{\mathbf{3}}(0,2,1) &= 1/2\, p((6,2,5) \mid f^{\varepsilon}_{-i}) + 4/3\, p((6,4,5) \mid f^{\varepsilon}_{-i}) \\ &= 19/12\, p((6,4,5) \mid f^{\varepsilon}_{-i}) \\ &> U_{\mathbf{3}}(0,2,0) = U_{\mathbf{3}}(0,2,2). \end{aligned}$$

the non-overstating strategy is the best reply to f^{ε}_{-i}. The same statement applies for voter **4**.

Similarly, one can deduce that for voters $i = \mathbf{5}, \mathbf{6}, \mathbf{7}$ casting ballot (0,0,2) is a best response against f^{ε}. Indeed, for voters **5**, **6** and **7**, the relevant contingencies are summarized by the vectors $(6,4,4)$ and $(6,6,4)$. Let us denote their probabilities by $p((6,4,4) \mid f^{\varepsilon}_{-i})$ and $p((6,6,4) \mid f^{\varepsilon}_{-i})$. Furthermore, given voter's best responses, we can write that $2p((6,4,4) \mid f^{\varepsilon}_{-i}] = p((6,6,4) \mid f^{\varepsilon}_{-i})$. Since

$$\begin{aligned} U_{\mathbf{5}}(0,0,2) &= 3/2\, p((6,4,4) \mid f^{\varepsilon}_{-i}) + 4/3\, p((6,6,4) \mid f^{\varepsilon}_{-i}) \\ &= 25/12\, p((6,6,4) \mid f^{\varepsilon}_{-i}) \\ &> U_{\mathbf{5}}(0,1,2), U_{\mathbf{5}}(0,2,2). \end{aligned}$$

the non-overstating strategy is the best reply to f^{ε}_{-i} and similarly for voters **6** and **7**.

Hence, $\{f^{\varepsilon}\}$ is a sequence of ε-perfect equilibria. Since f is the limit of f^{ε}, it is a perfect equilibrium in which voters' best responses are not overstating.

B Appendix: Proof of Proposition 3

The first step of the proof consists in showing that g is a mixed strategy equilibrium. To do so, we compute the probability, under g, of each pivot outcome a player can face and, from these probabilities, the expected utility derived from each undominated strategy.

Voters 1,2,3

Even though the best responses are explained for the voter **1**, the reasoning is analogous for voters **2** and **3**.

$p((4,4,6) \mid g_{-1}) = 1/3$
$p((4,5,6) \mid g_{-1}) = 1/3$
$p((4,6,6) \mid g_{-1}) = 1/3$.
From the pivot probabilities previously described, we have
$U_{\mathbf{1}}(2,0,0) = 25/9$
$U_{\mathbf{1}}(2,1,0) = 19/9$
$U_{\mathbf{1}}(2,2,0) = 13/9$.
which entails that $(2,0,0)$ is the unique best response for voter **1**.

Voters 4,5

Voter **4**'s best responses are analyzed, the reasoning being analogous for the voter **5**.

$p((6,2,5) \mid g_{-4}) = 1/3$
$p((6,3,5) \mid g_{-4}) = 1/3$
$p((6,4,5) \mid g_{-4}) = 1/3$.
From the pivot probabilities previously described, we have
$U_{\mathbf{4}}(0,2,0) = 1$
$U_{\mathbf{4}}(0,2,1) = 10/9$
$U_{\mathbf{4}}(0,2,2) = 1$.
implying that $(0,2,1)$ is the unique best response for voter **4**.

Voter 6

The most probable pivot outcomes faced by voter **6** are as follows
$p((6,4,4) \mid g_{-6}) = 1/3$
$p((6,5,4) \mid g_{-6}) = 1/3$
$p((6,6,4) \mid g_{-6}) = 1/3$
From the pivot probabilities previously described, we have
$U_{\mathbf{6}}(0,0,2) = 25/9$
$U_{\mathbf{6}}(0,1,2) = 19/9$
$U_{\mathbf{6}}(0,2,2) = 13/9.$
implying that $(0,0,2)$ is the unique best response for voter **6**.

Voter 7

The most probable pivot outcome faced by voter **7** is the event $(6,4,4)$. Due to her utility profile, voter **7** strictly prefers to use an undominated strategy and is indifferent among all of them: that is $(0,0,2),(0,1,2),(0,2,2)$. Hence, the mixed strategy g_7 is a best response.

The second step of the proof consists in showing that g is a perfect equilibrium. To do so, consider the following completely mixed strategy combination g^ε, where η_i denotes the mixed strategy of voter i which assigns equal probability to all his pure strategies.

$$\begin{aligned}
&i = \mathbf{1},\mathbf{2},\mathbf{3} \;\; g_i^\varepsilon = (1-27\varepsilon^2)(2,0,0) + 27\varepsilon^2\eta_i \\
&i = \mathbf{4},\mathbf{5} \;\; g_i^\varepsilon = (1-27\varepsilon^2)(0,2,1) + 27\varepsilon^2\eta_i \\
&i = \mathbf{6} \;\; g_i^\varepsilon = (1-\varepsilon-27\varepsilon^2)(0,0,2) + \varepsilon(0,1,2) + 27\varepsilon^2\eta_i, \\
&i = \mathbf{7} \;\; g_i^\varepsilon = g_7 + 27\varepsilon^2\eta_i.
\end{aligned}$$

It is easy to see that, for ε sufficiently close to zero, this is an ε-perfect equilibrium. Suppose all voters other than i choose the strategies prescribed by g. Since for ε going to zero, the probability of voter **6** to tremble towards $(0,1,2)$ is infinitely greater than the probability of any other mistake, it is enough to check that the limiting strategy is preferred to the other undominated strategy when either this mistake or no mistake at all occurs.

For voters **1** to **5**, the relevant contingency is the one described by the limiting strategy g. Indeed, as has been shown, their unique best response is the one depicted by g as when the trembles tends towards, they have a unique best response. For voter **6**, the same argument applies.

Finally, one can deduce that for voter $i = \mathbf{7}$ casting the mixed strategy ballot g_7 is a best response, against g^ε. Indeed, for voter **7**, the relevant contingency are summarized by the vectors $(6, 4, 4)$ and $(6, 5, 4)$. Let us denote their probabilities by $p((6, 4, 4) \mid f^\varepsilon_{-i})$ and $p((6, 6, 4) \mid f^\varepsilon_{-i})$. Since

$$\begin{aligned} U_{\mathbf{7}}(0, 0, 2) &= 3p((6, 4, 4) \mid f^\varepsilon_{-i}) + \ 3p((6, 5, 4) \mid f^\varepsilon_{-i}) \\ &= U_{\mathbf{7}}(0, 1, 2), U_{\mathbf{7}}(0, 2, 2). \end{aligned}$$

the mixed strategy g_7 is a best reply to g^ε_{-i}.

Hence, $\{g^\varepsilon\}$ is a sequence of ε-perfect equilibria. Since g is the limit of g^ε, it is a perfect equilibrium in which voters' best responses are not overstating.