ECOLE POLYTECHNIQUE

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

PARETO INDIVISIBLE ALLOCATIONS, REVEALED PREFERENCE AND DUALITY

Ivar EKELAND

Alfred GALICHON

December 2010

Cahier n° 2010-37

DEPARTEMENT D'ECONOMIE

Route de Saclay

91128 PALAISEAU CEDEX

(33) 1 69333033

http://www.enseignement.polytechnique.fr/economie/

mailto:chantal.poujouly@polytechnique.edu

Pareto indivisible allocations, revealed preference and duality

Ivar Ekeland (University of British Columbia)
Alfred Galichon (Ecole polytechnique)

First version is November 23, 2010.* The present version is of December 14, 2010.

Abstract

This paper exhibits a duality between the theory of Revealed Preference of Afriat and the housing allocation problem of Shapley and Scarf. In particular, it is shown that Afriat's theorem can be interpreted as a second welfare theorem in the housing problem. Using this duality, the revealed preference problem is connected to an optimal assignment problem, and a geometrical characterization of the rationalizability of experiment data is given. This allows in turn to give new indices of rationalizability of the data, and to define weaker notions of rationalizability, in the spirit of Afriat's efficiency index.

JEL codes: primary D11, secondary C60, C78.

Keywords: revealed preferences, Afriat's theorem, optimal assignments, indivisible allocations.

*Correspondence: Economics Department, Ecole polytechnique, 91128 Palaiseau, France. E-mail: alfred.galichon@polytechnique.edu. Support from Chaire EDF-Calyon "Finance and Développement Durable" and FiME, Laboratoire de Finance des Marchés de l'Energie (www.fime-lab.org) is gratefully acknowledged by both authors, and from Chaire Axa "Assurance et Risques Majeurs," Chaire FDIR "Socially Responsible Investments and Value Creation" by the second author. The authors thank John Geanakoplos for communicating them his paper "Afriat from MaxMin". This paper has benefited from insightful comments by Don Brown, Françoise Forges, Peter Hammond and Enrico Minelli.

1 Introduction

The "Revealed Preference" problem and the "Housing problem" are two classical problems in Economic Theory with for both a distinguished, but separate tradition. This paper is about connecting them, and exploit the connection to obtain new results on Revealed Preference.

- The Revealed Preference (RP) problem, posed by Samuelson at the end of the 1930's, was solved by Sidney Afriat in 1967. This classical problem asks whether, given the observation of n consumptions baskets and corresponding prices, one rationalize these consumptions as the consumption of a single representative consumer facing different prices.
- The "housing problem" was investigated in 1974 by Shapley and Scarf. Given an initial allocation of n houses to n individuals, and assuming individuals form preferences over houses and can trade houses, what is the core of the corresponding game? It is assumed that houses form no preferences over owners (in sharp contrast to the "Stable matching" problem of Gale and Shapley). In this setting, they showed non-emptiness of the core, as well as an algorithm to arrive to a core allocation: the method of "top-trading cycles", attributed to David Gale.

Although both problems have generated two well established and distinct literatures, it will turn out that these problems are in fact dual in a precise sense. As we shall argue, the traditional expenditure/utility duality from consumer theory extends to the setting of revealed preference, and it is possible to show that the problem of Pareto efficiency in the housing problem is dual to a generalization of Afriat's theorem proven by Fostel, Scarf and Todd (2004). In particular, we show that *Afriat's theorem can be interpreted as a second welfare theorem in the housing problem*. Once we have established equivalence of both problems, we shall use the aforementioned results to give new characterization of both problems in terms of an optimal allocation problem. This will give us a very simple and intuitive characterization of the efficient outcomes in the indivisible allocation problem, as well as natural indexes of rationalizability. We shall last use these results to provide a geometric interpretation of revealed preference theory, by showing that the problem of determining whether data can be rationalized or not can be recast as the problem of determining whether a particular point is on some part of the boundary of some convex body.

The literature on revealed preference in consumer demand traces back to Samuelson (1938), who formulated the problem and left it open. It was solved by Afriat (1967) using nontrivial combinatorial techniques. Diewert (1973) provided a Linear Programming proof and Varian (1982) an algorithmic solution. Fostel, Scarf and Todd (2004) provided alternative proofs.

Matzkin (1991) and Forges and Minelli (2009) extended the theory to nonlinear budget constraints. Geanakoplos (2006) gives a proof of Afriat's theorem using a minmax theorem. Although min-max formulations will appear in our paper, these are distinct from Geanakoplos', as it will be explained below. The literature on the indivisible allocation problem was initiated by Shapley and Scarf (1974), who formulated as the "housing problem" and gave an abstract characterization of the core. Roth et al (2004) study a related "kidney problem" and investigate mechanism design aspect. Finally, recent literature has extended revealed preference theory to classes of matching problems. Galichon and Salanié (2010) investigate the problem of revealed preferences in a matching game with transferable utility, with and without unobserved heterogeneity. Echenique, Lee and Shum (2010) investigate the problem of revealed preferences in games with and without transferable utility, and without unobserved heterogeneity.

This paper will be organized as follows. We first study Pareto efficient allocations, and show the connection with the generalized theory of revealed preferences. We deduce a novel characterization of data rationalizability in terms of an optimal matching problem, and a geometric interpretation of revealed preference in terms of convex geometry. We then move on to providing indexes of increasingly weaker forms of rationalizability.

2 Revealed preference and the housing problem

2.1 The Generalized Afriat's theorem

Assume as in Forges and Minelli (2009) that consumer has budget constraint $g_i(x) \leq 0$ in experiment i, and that x_i is chosen where $g_i(x_i) = 0$. This is a nonlinear generalization of Afriat (1969), in which $g_i(x_j) = x_j \cdot p_i - x_i \cdot p_i$. Forges and Minelli ask whether there exists a utility function $v(x)$ with appropriate properties such that

$$x_i \in \operatorname{argmax} \{x : g_i(x) \leq 0\}.$$

As in the original Afriat's paper, the first part of their proof necessitates the existence of a utility level v_j associated to good j such that

$$g_i(x_j) < 0 \text{ implies } v_j - v_i < 0. \quad (1)$$

Once the number v_i's have been determined, the second part of their proof consists in constructing a locally nonsatiated and continuous utility function $v(x)$ such that $v(x_j) = v_j$. The preference induced by v are said to rationalize the data in the sense that

$$i \in \operatorname{argmax}_j \{v_j : g_i(x_j) \leq 0\}.$$

The existence of real numbers v_i such that (1) holds is non-trivial and it turns out to be equivalent to a property of matrix $R_{ij} = g_i(x_j)$ called "cyclical consistency". This is done by appealing to an extension of Afriat's original theorem, proved by Fostel, Scarf and Todd (2004). This result can be slightly reformulated as follows:

Theorem 0 (Afriat's theorem)**.** *The following conditions are equivalent:*

*(i) The matrix R_{ij} satisfies **"cyclical consistency"**: for any cycle $i_1, ..., i_{p+1} = i_1$,*

$$\forall k,\ R_{i_k i_{k+1}} \leq 0 \textit{ implies } \forall k,\ R_{i_k i_{k+1}} = 0, \tag{2}$$

(ii) There exist numbers (v_i, λ_i), $\lambda_i > 0$, such that

$$v_j - v_i \leq \lambda_i R_{ij},$$

(iii) There exist numbers v_i such that

$$\begin{aligned} R_{ij} &\leq 0 \textit{ implies } v_j - v_i \leq 0, \textit{ and} \\ R_{ij} &< 0 \textit{ implies } v_j - v_i < 0. \end{aligned}$$

Proof. (i) implies (ii) is proven in Fostel, Scarf and Todd (2004). (ii) immediately implies (iii). We now show that (iii) implies (i). Consider a cycle $i_1, ..., i_{p+1} = i_1$, such that

$$\forall k,\ R_{i_k i_{k+1}} \leq 0.$$

Then by (iii) there exist numbers v_i such that

$$\begin{aligned} R_{ij} &\leq 0 \text{ implies } v_j - v_i \leq 0, \text{ and} \\ R_{ij} &< 0 \text{ implies } v_j - v_i < 0. \end{aligned}$$

thus one has $v_{i_{k+1}} - v_{i_k} \leq 0$ for all k, hence all the v_{i_k} are equal. Assume now that there is a k such that $R_{i_k i_{k+1}} < 0$. Then $v_{i_{k+1}} - v_{i_k} < 0$, which is a contradiction. Therefore,

$$\forall k,\ R_{i_k i_{k+1}} = 0,$$

which proves the cyclical consistency of matrix R, that is (i). ∎

2.2 Pareto efficient allocation of indivisible goods

We now turn to the problem of allocation of indivisible goods, which was initially studied by Shapley and Scarf (1974). Consider n indivisible goods (eg. houses) $j = 1, ..., n$ to be allocated to n individuals. Cost of allocating (eg. transportation cost) house j to individual i is c_{ij}. An allocation is a permutation σ of the set $\{1, .., n\}$ such that individual i gets house $j = \sigma(i)$. Let S be the set of such permutations. We assume for the moment that the initial allocation is given by the identity permutation: good i is allocated to individual i. The problem here is to decide whether this allocation is efficient in a Pareto sense.

If there are two individuals, say i_1 and i_2 that would both benefit from swapping houses (strictly for at least one), then this allocation is not efficient, as the swap would improve on the welfare of both individuals. Thus if allocation is efficient, then inequalities $c_{i_1 i_2} \leq c_{i_1 i_1}$ and $c_{i_2 i_1} \leq c_{i_2 i_2}$ cannot hold simultaneously unless they are both equalities. More generally, Pareto efficiency rules out the existence of exchange rings whose members would benefit (strictly for at least one) from a circular trade. In fact, we shall argue that *this problem is dual to the problem of Generalized Revealed Preferences*.

2.3 A dual interpretation of revealed preference

Let us now formalize the notion of efficient allocation in the previous discussion. Allocation $\sigma_0(i) = i$ is Pareto efficient if and only if for any $\sigma \in S$, inequalities

$$c_{i\sigma(i)} \leq c_{ii}$$

cannot hold simultaneously unless these are all equalities. By the decomposition of a permutation into cycles, we see that this definition is equivalent to the fact that for every "trading cycle" $i_1, ..., i_{p+1} = i_1$,

$$\forall k,\ c_{i_k i_{k+1}} \leq c_{i_k i_k} \text{ implies } \forall k,\ c_{i_k i_{k+1}} = c_{i_k i_k}$$

that is, introducing $R_{ij} = c_{ij} - c_{ii}$,

$$\forall k,\ R_{i_k i_{k+1}} \leq 0 \text{ implies } \forall k,\ R_{i_k i_{k+1}} = 0,$$

which is to say that allocation is efficient if and only if the matrix R_{ij} is cyclically consistent.

By the equivalence between (i) and (ii) in Theorem 0 above, we have the following statement:

Proposition 1. *In the housing problem, allocation $\sigma_0(i) = i$ is efficient if and only if*

$$\exists v_i \text{ and } \lambda_i > 0,\ v_j - v_i \leq \lambda_i R_{ij}. \quad \text{(PARETO)}$$

Before giving an interpretation of this result, we would like to understand the link between efficiency and equilibrium in the housing problem. Assume we start from allocation $\sigma_0(i) = i$, and we let people trade. Given a price system π where π_i is the price of house i, we assume that individual i can sell her house for price π_i, and therefore can afford any house j whose price π_j is less than π_i. Therefore, individual i's budget set B_i is the set of houses that sell to a price lower than his'

$$B_i = \{j : v_j \leq v_i\}$$

We can now define the notion of a competitive equilibrium in this setting. Allocation $\sigma \in S$ is an equilibrium supported by price system π if for each individual i, 1) house $\sigma(i)$ is weakly preferred by individual i among all houses that she can afford, while 2) house $\sigma(i)$ is strictly preferred by i among all houses she can strictly afford (i.e., that trade for prices strictly less than her house i). While condition 1) is a necessary requirement, refinement 2) is natural from a behavioral point of view as it implies that if i is indifferent between two houses, then she is going to choose the cheapest house of the two.

In particular, $\sigma_0(i) = i$ is a *No-Trade equilibrium* if there is a system of prices π, where π_j is the price of house j, such that: 1) whenever house j is affordable for individual i, then it is not strictly preferred by i to i's house, that is $\pi_j \leq \pi_i$ implies $c_{ij} \geq c_{ii}$, and 2) for any house j in the strict interior of i's budget set, i.e. that trades for strictly cheaper than house i, then individual i strictly prefers her own house i to house j, that is $\pi_j < \pi_i$ implies $c_{ij} > c_{ii}$.

Hence, $\sigma_0(i) = i$ is a No-Trade equilibrium supported by prices π when conditions (E1) and (E2) below are met, where:

(E1) if house j can be afforded by i, then individual i does not strictly prefer house j to house i, that is,

$$\pi_j \leq \pi_i \text{ implies } c_{ij} \geq c_{ii},$$

that is

$$\pi_j \leq \pi_i \text{ implies } R_{ij} \geq 0,$$

that is, yet equivalently:

$$R_{ij} < 0 \text{ implies } \pi_j > \pi_i.$$

(E2) if house j is (strictly) cheaper than house i, then individual i strictly prefers house i to house j, that is

$$\pi_j < \pi_i \text{ implies } c_{ij} > c_{ii},$$

that is

$$\pi_j < \pi_i \text{ implies } R_{ij} > 0,$$

that is, yet equivalently:

$$R_{ij} \leq 0 \text{ implies } \pi_j \geq \pi_i.$$

Proposition 2. *In the housing problem, allocation $\sigma_0(i) = i$ is a No-Trade equilibrium supported by prices π if and only if*

$$\begin{aligned} R_{ij} &< 0 \text{ implies } \pi_j > \pi_i, \text{ and} \\ R_{ij} &\leq 0 \text{ implies } \pi_j \geq \pi_i. \end{aligned} \qquad \text{(EQUILIBRIUM)}$$

But (EQUILIBRIUM) is exactly formulation (iii) of Theorem 0 with $\pi_i = -v_i$. By Theorem 0, we know that this statements is equivalent to statement (PARETO). Hence, we get an interpretation of *the Generalized Afriat's Theorem as a second welfare theorem*

$$\text{(EQUILIBRIUM)} \iff \text{(PARETO)},$$

which we summarize in the following proposition:

Proposition 3. *In the allocation problem of indivisible goods, Pareto allocations are no-trade equilibria supported by prices, and conversely, no-trade equilibria are Pareto efficient.*

This is a "dual" interpretation of revealed preference, where v_i (utilities in generalized RP theory) become budgets here, and c_{ij} (budgets in generalized RP theory) become utilities here. To summarize this duality, let us give the following table:

	Revealed prefs.	Pareto indiv. allocs.
setting	consumer demand	allocation problem
budget sets	$\{j : c_{ij} \leq c_{ii}\}$	$\{-v : -v \leq -v_i\}$
cardinal utilities to j	v_j	$-c_{ij}$
# of consumers	one, representative	n, $i \in \{1, ..., n\}$
# of experiments	n	one
goods	divisible	indivisible
unit of c_{ij}	dollars	utils
unit of v_i	utils	dollars
interpretation	Afriat's theorem	Welfare theorem

2.4 A characterization of rationalizability

The connection of both the revealed preference problem and the housing problem with assignment problems is clear; we shall now show that there is a very useful connection with the optimal assignment problem (formulated as a Linear Programming problem by Dantzig in the 1930s; see Shapley and Shubik 1971 for a game-theoretic interpretation), which we recall here.

In this problem, one seeks the assignment $\sigma \in S$ which minimizes the utilitarist welfare loss computed as the sum of the individual costs, setting weight one to each individuals. This problem is therefore

$$\min_{\sigma \in S} \sum_{i=1}^{n} c_{i\sigma(i)}.$$

By the standard Linear Programming duality of the optimal assignment problem (Dantzig 1939; Shapley-Shubik 1971)

$$\min_{\sigma \in S} \sum_{i=1}^{n} c_{i\sigma(i)} = \max_{u_i + v_j \leq c_{ij}} \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j, \tag{3}$$

where S is the set of permutations of $\{1, ..., n\}$. It is well-known that for a $\sigma_0 \in S$ solution to the optimal assignment problem, there is a pair (u, v) solution to the dual problem such that

$$\begin{array}{rcl} u_i + v_j & \leq & c_{ij} \\ \text{if } j = \sigma_0 (i), & & \text{then } u_i + v_j = c_{ij}. \end{array}$$

With this in mind, we have the following novel characterization of rationalizability of the data, which extends Theorem 0:

Theorem 4. *In the revealed preference problem, the data are rationalizable if and only if there exist weights* $\lambda_i > 0$ *such that*

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0,$$

that is

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i g_i \left(x_{\sigma(i)}\right) = \sum_{i=1}^{n} \lambda_i g_i (x_j). \tag{4}$$

Proof of Theorem 4. One has

(4) $\iff \exists \lambda_i > 0, \ \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0$

$\iff \exists \lambda_i > 0, \ \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}$ is reached for $\sigma = Id$

$\iff \exists \lambda_i > 0, u, v \in \mathbb{R}^n$

$$\begin{array}{rcl} u_i + v_j & \leq & \lambda_i R_{ij} \\ u_i + v_i & = & 0 \end{array}$$

$\iff \exists \lambda_i > 0, v \in \mathbb{R}^n$

$$v_j - v_i \leq \lambda_i R_{ij},$$

which is (ii) in Theorem 0. ∎

The previous result leads to the following remark. Introduce the set in $\mathbb{R}^d$

$$\mathcal{F} = conv\left(\left(g_i\left(x_{\sigma(i)}\right)\right)_{i=1,\dots,n} : \sigma \in S\right).$$

this set is a convex polytope. Note that

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i g_i\left(x_{\sigma(i)}\right) = \min_{C \in \mathcal{F}} \sum_{i=1}^{n} \lambda_i g_i\left(x_i\right).$$

We have the following geometric characterization of the fact that the data are rationalizable:

Proposition 5. *The data are rationalizable if and only if* $C_i^0 = g_i(x_i)$ *is a vertex of* $F = conv\left(\left(g_i\left(x_{\sigma(i)}\right)\right)_{i=1,\dots,n} : \sigma \in S\right)$ *with a componentwise positive vector in the normal cone.*

Proof. Introduce $\mathcal{W}(\lambda) = \min_{C \in \mathcal{F}} \sum_{i=1}^{n} \lambda_i C_i$. This concave function is the support function of C. $C_i^0 = c_{i\sigma_0(i)}$ is a vertex of $\mathcal{F}$ with a componentwise positive vector in the normal cone if and only C^0 is in the superdifferential of $\mathcal{W}$ at such a vector λ. This holds if and only if $\mathcal{W}(\lambda) = \sum_{i=1}^{n} \lambda_i C_i^0$. ∎

3 Strong and weak rationalizability

3.1 Indices of rationalizability

Rationalizability of the data by a single representative consumer's, as tested by Afriat's inequalities, is most often rejected in practice. Hence it is of interest to introduce measures of "how close" the data is from being rationalizable. This is in the spirit of Afriat's original efficiency index, which is the largest $e \leq 1$ such that $R_{ij}^e = R_{ij} + (1-e)\,b_i$ is cyclically consistent, where $b_i > 0$ is a fixed vector with positive components. In Afriat's setting, $R_{ij} = x_j \cdot p_i - x_i \cdot p_i$, and $R_{ij}^e = x_j \cdot p_i - e x_i \cdot p_i$, so $b_i = x_i \cdot p_i$. In the same spirit, we shall introduce indices that will measure how far the dataset is from being rationalizable. The indices we shall build are connected to the dual interpretation of revealed preferences we have outlined above. Our measures of departure from rationalizability will be connected to measures of departure from Pareto efficiency in that problem. One measure of departure from efficiency is the welfare gains that one would gain from moving to the efficient frontier; this is interpretable as Debreu's (1951) *coefficient of resource utilization*[1], in the case of a convex economy. This is only an analogy, as

[1] We thank Don Brown for suggesting this interpretation to us.

we are here in an indivisible setting, but this exactly the idea we shall base the construction of our indices on.

It seems natural to introduce index A as

$$A = \max_{\lambda \in \Delta} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}$$

where $\Delta = \{\lambda \geq 0, \sum_{i=1}^{n} \lambda_i = 1\}$ is the simplex of $\mathbb{R}^n$. Indeed, we have

$$A \leq 0,$$

and by compacity of Δ, equality holds if and only if there exist $\lambda \in \Delta$ such that

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0.$$

Of course, this differs from our characterization of rationalizability in 4, as there the weights λ_i's need to be all positive, not simply nonnegative. It is easy to construct examples where (4) holds with some zero λ_i's and $\sigma_0(i) = i$ is not efficient. For example, in the housing problem, if individual $i = 1$ has his most preferred option, then $\lambda_1 = 0$ and all the other λ_i's are zero, and $A = 0$, thus (4) holds. However, allocation may not be Pareto because there may be inefficiencies among the rest of the individuals.

Hence imposing $\lambda > 0$ is crucial. Fortunately, it turns out that one can restrict Δ to a subset which is convex, compact and away from zero, as shown in the next lemma.

Lemma 6. *There is $\varepsilon > 0$ (dependent only on matrix R) such that the λ_i's in Theorem 4 above (if they exist) can be chosen such that*

$$\begin{cases} \lambda_i \geq \varepsilon \text{ for all } i, \\ \sum_{i=1}^{n} \lambda_i = 1. \end{cases}$$

Proof. Standard construction (see (*Fostel et al. (2004)*)) of the λ_i's and the v_i's provides a deterministic procedure that returns strictly positive $\lambda_i \geq 1$ within a finite and bounded number of steps, with only the entries of R_{ij} as input; hence λ, if it exists, is bounded, so there exists M depending only on R such that $\sum_{i=1}^{n} \lambda_i \leq M$. Normalizing λ so that $\sum_{i=1}^{n} \lambda_i = 1$, one sees that one can choose $\varepsilon = 1/M$. ∎

We denote Δ_ε the set of such vectors λ, and we recall that Δ is the set of λ such that $\lambda_i \geq 0$ for all i and $\sum_{i=1}^{n} \lambda_i = 1$. Recall $R_{ij} = c_{ij} - c_{ii}$, and introduce

$$A^* = \max_{\lambda \in \Delta_\varepsilon} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)},$$

so that we have

$$A^* \leq 0$$

and equality if and only if the data are rationalizable (as in this case there exist $\lambda_i > 0$ such that the characterization of rationalizability in 4 is met). Further, as $\Delta_\varepsilon \subset \Delta$, one gets

$$\underbrace{\max_{\lambda \in \Delta_\varepsilon} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{A^*} \leq \underbrace{\max_{\lambda \in \Delta} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{A} \leq 0.$$

The max-min formulation for index A leads naturally to introduce a new index which comes from the dual min-max program

$$\begin{aligned} B &= \min_{\sigma \in S} \max_{\lambda \in \Delta} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} \\ &= \min_{\sigma \in S} \max_{i \in \{1,\dots,n\}} R_{i\sigma(i)} \end{aligned}$$

Note that by weak duality $\max\min \leq \min\max$, so $A \leq B$, and further, we have

$$B \leq \max_{i \in \{1,\dots,n\}} R_{ii} = 0.$$

Therefore we have

$$\underbrace{\max_{\lambda \in \Delta_\varepsilon} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{A^*} \leq \underbrace{\max_{\lambda \in \Delta} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{A} \leq \underbrace{\min_{\sigma \in S} \max_{\lambda \in \Delta} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{B} \leq 0.$$

We shall come back to the interpretation of $A^* = 0$, $A = 0$ and $B = 0$ as stronger or weaker forms of rationalizability of the data. Before we do that, we summarize the above results.

Proposition 7. *We have:*

(i) $A^* = 0$ *if and only if there exist scalars* v_i *and weights* $\lambda_i > 0$ *such that*

$$v_j - v_i \leq \lambda_i R_{ij}.$$

(ii) $A = 0$ *if and only if there exist scalars* v_i *and weights* $\lambda_i \geq 0$*, not all zero, such that*

$$v_j - v_i \leq \lambda_i R_{ij}.$$

(iii) $B = 0$ *if and only if*

$$\forall \sigma \in S,\ \exists i \in \{1, ..., n\} : R_{i\sigma(i)} \geq 0.$$

(iv) One has

$$A^* \leq A \leq B \leq 0.$$

Proof. (i) follows from Lemma 6. To see (ii), note that $A = 0$ is equivalent to the existence of $\lambda \in \Delta$ such that $\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0$, and the rest follows from Theorem 4. (iii) is equivalent to the fact that for all $\sigma \in S$, $\max_{i \in \{1,...,n\}} R_{i\sigma(i)} \geq 0$, that is for all $\sigma \in S$, there exists $i \in \{1, ..., n\}$ such that $R_{i\sigma(i)} \geq 0$. The inequalities in (iv) were explained above. ∎

In an unpublished manuscript ((*Geanakoplos (2006)*)) that he kindly accepted to communicate to us, John Geanakoplos introduces the following minmax problem, which in our notations can be defined as

$$G = \max_{\lambda \in \Delta} \min_{\sigma \in C} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}$$

where $C \subset S$ is the set of permutations that have only one cycle, ie. such that there exist a cycle $i_1, ..., i_{p+1} = i_1$ such that $\sigma(i_k) = i_{k+1}$ and $\sigma(i) = i$ if $i \notin \{i_1, ..., i_p\}$. As $C \subset S$ one has

$$A \leq G \leq 0.$$

Geanakoplos uses index G and von Neuman's minmax theorem in order to provide an interesting alternative proof of Afriat's theorem. However, it seems that index G is not directly connected with the assignment problem.

3.2 Interpretation of indexes A^*, A, B

As seen above, the index A^* was constructed so that $A^* = 0$ if and only if the data are rationalizable. The indexes A and B will both be equal to 0 if the data are rationalizable; hence $A < 0$ or $B < 0$ imply that the data is not rationalizable; however the converse is not true, so these indexes can be interpreted as measures of weaker form of rationalizability. Hence we would like to give a meaningful interpretation of the situations where $A = 0$ and $B = 0$. It turns out that $A = 0$ is equivalent to the fact that a subset of the observations can be rationalized, the subset having to have a property of *coherence* that we now define. $B = 0$ is equivalent to the fact that one cannot partition the set of observations into *increasing cycles*, a notion we will now define.

Throughout this subsection it will be assumed that no individual is indifferent between two distinct consumptions for the direct revealed preference relation, that is:

Assumption A. *In this subsection, R is assumed to verify $R_{ij} \neq 0$ for $i \neq j$.*

We first define the notion of coherent subset of observations.

Definition 8 (Coherent subset). *In the revealed preference problem, a subset of observations included in* $\{1, ..., n\}$ *is said to be coherent when* $i \in I$ *and* i *directly revealed preferred to* j *implies* $j \in I$*. Namely,* I *is coherent when*

$$i \in I \text{ and } R_{ij} < 0 \text{ implies } j \in I.$$

In particular, $\{1, ..., n\}$ is coherent; any subset of observations where each observation is directly revealed preferred to no other one is also coherent. Next, we define the notion of increasing cycles.

Definition 9 (Increasing cycles). *A cycle* $i_1, ..., i_{p+1} = i_1$ *is called* increasing *when each observation is strictly directly revealed preferred to its predecessor. Namely, cycle* $i_1, ..., i_{p+1} = i_1$ *is increasing when*

$$R_{i_k i_{k+1}} < 0 \text{ for all } k \in \{1, ..., p\}.$$

Of course, the existence of an increasing cycle implies that the matrix R is not cyclically consistent, hence it implies in the revealed preference problem that the data are not rationalizable. It results from the definition that an increasing cycle has length greater than one.

We now state the main result of this section, which provides an economic interpretation for the indexes A^*, A and B.

Theorem 10. *We have:*
(i) $A^* = 0$ *iff the data are rationalizable,*
(ii) $A = 0$ *iff a coherent subset of the data is rationalizable,*
(iii) $B = 0$ *iff there is no partition of* $\{1, ..., n\}$ *in increasing cycles.*
and (i) implies (ii), which implies (iii).

Proof. (i) was proved in Theorem 4 above.

Let us show the equivalence in (ii). From Proposition 7 $A = 0$ is equivalent to the existence of $\exists \lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ and $v \in \mathbb{R}^n$ such that

$$v_j - v_i \leq \lambda_i R_{ij},$$

so defining I as the set of $i \in \{1, ..., n\}$ such that $\lambda_i > 0$, this implies a subset I of the data is rationalizable. We now show that I is coherent. Indeed, for any two k and l not in I and i in I, one has $v_k = v_{l'} \geq v_i$; thus if $i \in I$ and $R_{ij} < 0$, then $v_j < v_i$, hence $j \in I$, which show that I is coherent.

Conversely, assume a coherent subset of the data I is rationalizable. Then there exist $(u_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ such that $\mu_i > 0$ and

$$u_j - u_i \leq \mu_i R_{ij}$$

for $i, j \in I$. Complete by $u_i = \max_{k\in I} u_k$ for $i \notin I$, and introduce $\tilde{R}_{ij} = 1_{\{i\in I\}} R_{ij}$. One has $\tilde{R}_{ij} < 0$ implies $i \in I$ and $R_{ij} < 0$ hence $j \in I$ by the coherence property of I, thus $u_j - u_i < 0$. Now $\tilde{R}_{ij} = 0$ implies either $i \notin I$ or $R_{ij} = 0$ thus $i = j$; in both cases, $u_j \leq u_i$. Therefore, one has

$$\begin{aligned} \tilde{R}_{ij} &< 0 \text{ implies } u_j - u_i < 0, \text{ and} \\ \tilde{R}_{ij} &\leq 0 \text{ implies } u_j - u_i \leq 0. \end{aligned}$$

Hence by Theorem 0, there exist v_i and $\bar{\lambda}_i > 0$ such that

$$v_j - v_i \leq \bar{\lambda}_i \tilde{R}_{ij}$$

and defining $\lambda_i = \bar{\lambda}_i 1_{\{i\in I\}}$, one has

$$v_j - v_i \leq \lambda_i R_{ij}$$

which is equivalent to $A = 0$.

(iii) Now Proposition 7 implies that $B < 0$ implies that there is $\sigma \in S$ such that $\forall i \in \{1, ..., n\}$, $R_{i\sigma(i)} < 0$. The decomposition of σ in cycles gives a partition of $\{1, ..., n\}$ in increasing cycles.

(iv) The implication $(i) \Rightarrow (ii) \Rightarrow (iii)$ results from inequality $A^* \leq A \leq B \leq 0$. ∎

3.3 Discussion and examples

Two observations

First, note that with two observations ($n = 2$), the general form of matrix R is

$$R = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

We shall first see that with two observations $n = 2$, $A^* < A$ in general but $A = B$. This comes from the fact that there is only one cycle that does not leave $\{1, 2\}$ invariant. Consider

$$R = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

and assume both a and b are not zero.

Proposition 11. *With two observations ($n = 2$), one can set*

$$\begin{aligned} \varepsilon &= \min\left(\frac{1}{2}; \frac{\max\{a,b,0\}}{\max\{a,b\} - \min\{a,b\}}\right) \text{ if } a \neq b \\ \varepsilon &= 1/2 \text{ otherwise} \end{aligned}$$

and one has

$$\begin{aligned} A^* &= \min\{0, (1-\varepsilon)\max\{a,b\} + \varepsilon \min\{a,b\}\} \\ A &= B = \min\{0, \max\{a,b\}\} \end{aligned}$$

Hence, $A^* = 0$ if and only if $a \leq 0$ and $b \leq 0$ cannot hold at the same time; $A = B = 0$ if and only if $a < 0$ and $b < 0$ cannot hold at the same time.

Proof. It is easily verified that

$$A = B = \min\{0, \max\{a,b\}\}.$$

Similarly, as $\varepsilon \leq 1/2$, $\max_{\lambda \in \Delta_\varepsilon}(\lambda_1 a + \lambda_2 b) = (1-\varepsilon)\max(a,b) + \varepsilon \min(a,b)$, so $A^* = \min\{0, (1-\varepsilon)\max\{a,b\} + \varepsilon \min\{a,b\}\}$. But with this choice of ε, it can be seen that $\max\{a,b\} \geq 0$ implies $(1-\varepsilon)\max\{a,b\} + \varepsilon \min\{a,b\} \geq 0$, the converse holding always. Hence one can set ε as announced. ∎

We shall now give some examples with $n = 2$. Consider

$$R = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

then

$$\begin{aligned} A^* &= -\varepsilon \\ A &= B = 0. \end{aligned}$$

In other words, the data is not rationalizable, but coherent subset $\{2\}$ is; and it is impossible to partition the data in increasing cycle, as the only candidate would be $\sigma(i) = i + 1 (\bmod 2)$, but that cycle is not increasing as $R_{21} = 0$.

Next, consider

$$R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

then

$$A^* = A = B = -1,$$

so the data is not rationalizable, even in the weakest sense as cycle $\sigma(i) = i + 1 (\bmod 2)$ is increasing.

Finally, consider

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

here, the data is rationalizable, and

$$A^* = A = B = 0.$$

Three observations

We saw above that with two observations, we have in general $A^* \leq A$ can hold strict but $A = B$. We now show that with three observations, inequality $A \leq B$ can hold strict as well. Introduce

$$c = \begin{pmatrix} 0 & -1 & -1 \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

We have

$$\begin{aligned} A &= -0.4 \\ B &= 0; \end{aligned}$$

indeed,

$$B = \min_{\sigma \in S} \max_i R_{i\sigma(i)}$$

but we can rule out the σ's with a fixed point, so we are left with $\sigma = (231)$ and (312); but $\max_i c_{i\sigma(i)}$ on these permutation is worth 2; hence $B = 0$. To compute A, one may use a Linear Programming formulation for A which results from the LP formulation of the optimal assignment problem (3):

$$A = \min \phi$$

subject to constraints

$$\begin{aligned} \phi - \sum_j \pi_{ij} c_{ij} &\geq 0 \, \forall i \\ \sum_j \pi_{ij} &= 1 \, \forall i \\ \sum_i \pi_{ij} &= 1 \, \forall j \\ \pi_{ij} &\geq 0 \, \forall ij \end{aligned}$$

A LP solver returns $A = -0.4$. This means that neither the entire dataset, nor a coherent subset of the data is rationalizable (as $A < 0$), but there is no partition of $\{1, 2.3\}$ into increasing cycles (as $B = 0$).

4 Concluding remarks

To conclude, we make a series or remarks.

First, we have shown how our dual interpretation of Afriat's theorem in its original Revealed Preference context as well as in a less traditional interpretation of efficiency in the housing problem could shed new light and give new tools for both problems. It would be interesting to undestand if there is a similar duality for problems of revealed preferences in matching games, recently investigated by Galichon and Salanié (2010) and Echenique, Lee and Shum (2010).

Also, we argue that it makes sense to investigate "weak rationalizability" (ie. $A = 0$ or $B = 0$) instead of "strong rationalizability" (ie. $A^* = 0$), or equivalently, it may make sense to allow some λ_i's to be zero. In the case of $A = 0$, recall the λ_i's are interpreted in Afriat's theory as the Lagrange multiplier of the budget constraints. Allowing for $\lambda = 0$ corresponds to excluding wealthiest individuals as outliers. It is well-known that when taken to the data, strong rationalizability is most often rejected. It would be interesting to test econometrically for weak rationalizability, namely whether $A = 0$.

$B < 0$ is a very strong measure of nonrationalizability, as $B < 0$ means that one can find a partition of the observation set into increasing cycles, which seems a very strong violations of the Generalized Axiom of Revealed Preference. But this may arise in some cases, especially with a limited number of observations.

Finally, we emphasize on the fact that indexes A^*, A and B provide a measure of how close the data is from being rationalizable, in the spirit of Afriat's efficiency index. It is clear for every empirical researcher that the relevant question about revealed preference in consumer demand is not whether the data satisfy GARP, it is how much they violate it. These indexes are an answer to that question. Also the geometric interpretation of rationalizable is likely to provide useful insights for handling unobserved heterogeneity. We plan to investigate this question in further research.

References and Notes

Afriat (1967). Afriat, S. (1967), "The construction of a Utility Function from Demand Data", *International Economic Review* 8, 67-77.

Diewert (1973). Diewert, W. (1973), "Afriat and Revealed Preference Theory", *Review of Economic Studies* 40, 419-425.

Echenique et al. (2010). Echenique, F. Lee, S.M., and Shum, M. (2010) "Aggregate Matchings", unpublished manuscript.

Debreu (1951). Debreu, G. (1951), "The Coefficient of Resource Utilization," *Econometrica*, 19, 273-292.

Forges and Minelli (2009). Forges, F. and Minelli, E., "Afriat's Theorem for General Budget Sets," *Journal of Economic Theory* 144 (2009) 135–145.

Fostel et al. (2004). Fostel, A., H. Scarf and M. Todd (2004), "Two New Proofs of Afriat's Theorem", *Economic Theory* 24, 211-219.

Galichon and Salanié (2010). Galichon, A., and Salanié, B. (2010) "Matchings with Trade-Offs: Revealed preferences over Competing Characteristics," unpublished manuscript.

Geanakoplos (2006). Geanakoplos, J. (2006), "Afriat from MaxMin," unpublished manuscript, private communication.

Matzkin (1991). Matzkin, R. (1991), "Axioms of Revealed Preferences for Nonlinear Choice Sets", *Econometrica* 59, 1779-1786.

Roth et al. (2004). Roth, A.E., T. Sönmez, and U. Ünver (2004) "Kidney exchange," *Quarterly Journal of Economics*, 119, 457–488.

Samuelson (1938). Samuelson, P. (1938), "A Note on the Pure Theory of Consumer's Behavior", *Economica* NS 5, 61-71.

Shapley and Scarf (1974). Shapley, L. and H. Scarf (1974) "On cores and indivisibility," *Journal of Mathematical Economics*, 1, 23–37.

Shapley and Shubik (1972). Shapley, L. and Shubik, M. (1972) "The assignment game I: the core," *International Journal of Game Theory* 1, pp. 111–130.

Varian (1982). Varian, H. (1982), "The Nonparametric Approach to Demand Analysis", *Econometrica* 50, 945-973.