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# DUAL THEORY OF CHOICE WITH MULTIVARIATE RISKS (1)

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ABSTRACT. We propose a multivariate extension of Yaari’s dual theory of choice under risk. We show that a decision maker with a preference relation on multidimensional prospects that preserves first order stochastic dominance and satisfies comonotonic independence behaves as if evaluating prospects using a weighted sum of quantiles. Both the notions of quantiles and of comonotonicity are extended to the multivariate framework using optimal transportation maps. Finally, risk averse decision makers are characterized within this framework and their local utility functions are derived. Applications to the measurement of multi-attribute inequality are also discussed.

**Keywords:** risk, rank dependent utility theory, multivariate comonotonicity, optimal transportation, multi-attribute inequality, Gini evaluation functions.

**JEL subject classification:** D63, D81, C61

## INTRODUCTION

In his seminal paper [34], Menahem Yaari proposed a theory of choice under risk, which he called “dual theory of choice,” where risky prospects are evaluated using a weighted sum of quantiles. The resulting utility is less vulnerable to paradoxes such as Allais’ celebrated paradox [1]. The main ingredients in Yaari’s representation are the preservation of first order stochastic dominance and insensitivity to hedging of comonotonic prospects. Both

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properties have strong normative and behavioral appeal once it is accepted that decision makers care only about the distribution of risky prospects. The preservation of stochastic dominance is justified by the fact that decision makers prefer risky prospects that yield higher values in all states of the world, whereas comonotonicity captures the decision maker’s insensitivity to hedging comonotonic prospects. The dual theory has been used extensively as an alternative to expected utility in a large number of contexts. The main drawback of the dual theory is that it does not properly handle the case in which the prospects of consumptions of different natures are not perfect substitutes. The assumption of law invariance (called neutrality in [34]) of the decision functional is easier to substantiate when several dimensions of the risk are considered in the decision functional.

To handle these situations, we need to be able to express utility derived from monetary consumption in different *numéraires*, which is easily done with Expected Utility theory, but so far not covered by Yaari’s dual theory. Indeed, the latter applies only to risky prospects defined as univariate random variables, thereby ruling out choice among multidimensional prospects which are not perfect substitutes for each other, such as risks involving both a liquidity and a price risk, collection of payments in different currencies, payments at different dates, prospects involving different goods of different natures such as consumption and environmental quality, etc. [33] proposes a multivariate version of his dual theory, but it involves independence of the risk components and an axiom of separability (Axiom A in [33]), which essentially removes the multidimensional nature of the problem.

We propose to remove this constraint with a multivariate extension of the dual theory to risky prospects defined as random vectors that is applicable as such to the examples listed above. The main challenge in this generalization is the definition of quantile functions and comonotonicity in the multivariate setting. Another challenge is to preserve the simplicity of the functional representing preferences, so that they can be parameterized and can be computed as efficiently as in the univariate case. Both challenges are met with an appeal to optimal transportation maps that allow for the definition of “generalized quantiles,” their efficient computation, and the extension of comonotonicity as a notion of distribution free perfect correlation. There are plenty of ways of extending the notion of comonotonicity to a multivariate framework consistently with the univariate definition. Our proposed extension has the added property of preserving the equivalence between comonotonicity and Pareto

efficiency of allocations (see [15] for the original result and [6] for the multivariate extension). With these notions of quantiles and comonotonicity in hand, we give a representation of a comonotonic independent preference relation as a weighted sum of generalized quantiles. The main difference between the univariate case and the multivariate case is that comonotonicity and generalized quantiles are defined with respect to an objective reference distribution, which features in the representation. The reference distribution is shown to be equal to the distribution of the stochastic discount factor of a risk averse Yaari decision maker and, as a result, it is equal to the distribution of equilibrium prices in an economy with at least one risk averse Yaari decision maker.

We then turn to the representation of risk averse decision makers's preferences within this theory. Risk aversion is defined in the usual way as a preference for less risky prospects, where the notion of increasing risk is suitably generalized to multivariate risky prospects. We show, again in a direct generalization of the univariate case, that risk aversion is characterized by a special form of the quantile weights defined above: risk averse decision makers give more weight to low outcomes (low quantiles) and less weight to high outcomes (high quantiles). As a result, given the reference distribution with respect to which comonotonicity is defined, risk averse decision makers are characterized by further simple restrictions on their utility functionals, which makes this model as simple and as tractable as expected utility. A further advantage of our decision functional is the simple characterization of the local utility function and its close relation to the multivariate quantile function.

The risk averse Yaari decision functional is a version of the Weymark social evaluation function (in [31]) with a continuous state space. Indeed, the formal equivalence between the evaluation of risky prospects and the measurement of inequality noted in [2] and [14] allows us to draw implications of our theory for the measurement of inequality of allocations of multiple attributes, such as consumption, education, environment quality etc... Seen as a social evaluation function, our decision functional provides a compromise between the approach of [11] and [29] in that it allows a flexible attitude to correlations between attributes without necessarily imposing correlation aversion and thereby circumventing the Bourguignon critique.

The paper is organized as follows. The next section gives a short exposition of the dual theory. The following section develops the generalized notion of comonotonicity that is

necessary for the multivariate extension, which is given in Section 3. Finally, risk aversion is characterized in section 4, application to multi-attribute inequality measurement is discussed in section 5 and the last section concludes.

**Notation and basic definitions.** Let  $(S, \mathcal{F}, \mathbb{P})$  be a non-atomic probability space. Let  $X : S \rightarrow \mathbb{R}^d$  be a random vector. We denote the probability distribution of  $X$  by  $\mathcal{L}_X$ .  $\mathbb{E}$  is the expectation operator with respect to  $\mathbb{P}$ . For  $x$  and  $y$  in  $\mathbb{R}^d$ , denote  $x \cdot y$  the standard scalar product of  $x$  and  $y$ , and  $\|x\|^2$  the Euclidian norm of  $x$ . We denote by  $X =_d \mathcal{L}_X$  the fact that the distribution of  $X$  is  $P$  and by  $X =_d Y$  the fact that  $X$  and  $Y$  have the same distribution. The *equidistribution class* of  $X =_d \mathcal{L}_X$ , denoted indifferently  $\text{equi}(\mathcal{L}_X)$  or  $\text{equi}(X)$ , is the set of random vectors with distribution with respect to  $\mathbb{P}$  equal to  $\mathcal{L}_X$  (reference to  $\mathbb{P}$  will be implicit unless stated otherwise).  $F_X$  denotes the cumulative distribution function of distribution  $\mathcal{L}_X$ .  $Q_X$  denotes its quantile function. In dimension 1, this is defined for all  $t \in [0, 1]$  by  $Q_X(t) = \inf_{x \in \mathbb{R}} \{\Pr(X \leq x) > t\}$ . In larger dimensions, it is defined in Definition 3 of Section 2.1 below. We call  $L_d^2$  the set of random vectors  $X$  in dimension  $d$  such that  $\mathbb{E}[\|X\|^2] < +\infty$ . We denote by  $\mathcal{D}$  the subset of  $L_d^2$  containing random vectors with a density relative to Lebesgue measure. For a convex lower semi-continuous function  $V : \mathbb{R}^d \mapsto \mathbb{R}$ , we denote by  $\nabla V$  its gradient (equal to the vector of partial derivatives).

## 1. DUAL THEORY OF DECISION UNDER RISK

In this section, we first revisit Yaari's "Dual theory of choice under risk" exposed in the eponymous paper [34]. As in the latter, we consider a problem of choice among risky prospects as modeled by random variables defined on an underlying probability space. The risky prospect  $X$  is interpreted as a gamble or a lottery that a decision maker might consider holding and the realizations of  $X$  are interpreted as payments.

**1.1. Representation.** We suppose that the decision maker is characterized by a preference relation  $\succsim$  on the set of risky prospects.  $X \succsim Y$  indicates that the decision maker prefers prospect  $X$  to prospect  $Y$ ,  $X \succ Y$  stands for  $X \succsim Y$  and not  $Y \succsim X$ , whereas  $X \sim Y$  stands for  $X \succsim Y$  and  $Y \succsim X$ . We first introduce the set of axioms satisfied by the preference relation that were proposed by Yaari in [34].

**Axiom 1.** *The preference relation  $\succsim$  is reflexive, transitive, connected and continuous relative to the topology of weak convergence.*

**Axiom 2.** *The preference  $\succsim$  preserves first order stochastic dominance in the sense that if prospect  $X$  first order stochastically dominates prospect  $Y$ , then  $X \succsim Y$ , and if  $X$  strongly first order stochastically dominates prospect  $Y$ , then  $X \succ Y$ .*

**Axiom 3.** *If  $X, Y, Z$  are pairwise comonotonic prospects, then for any  $\alpha \in [0, 1]$ ,  $X \succsim Y$  implies  $\alpha X + (1 - \alpha)Z \succsim \alpha Y + (1 - \alpha)Z$ .*

With the first axiom (which corresponds to Axioms A2 and A3 in Yaari [34]), we take the standard notion of preference as a continuous pre-order (reflexive and transitive binary relation) which is connected. Then,  $\succsim$  can be represented by a continuous real valued function  $\gamma$  in the sense that  $X \succsim Y$  if and only if  $\gamma(X) \geq \gamma(Y)$ .

The second axiom is generally considered uncontroversial in a law invariant setting. Note that since two equidistributed prospects first order stochastically dominate one another, it follows that  $X =_d Y$  implies  $X \sim Y$ . Thus this axiom implies the law invariance of the preference relation, or what [34] calls *neutrality*. Neutrality can be interpreted as the fact that the decision maker is indifferent to relabelings of the states of the world. Once neutrality is accepted, then Axiom 2 is uncontroversial as it is equivalent to requiring that the decision maker prefers prospects that yield a higher value in every state of the world. We shall see below that with suitable extensions of the concepts of monotonicity and stochastic dominance, this axiom remains uncontroversial in the multivariate extension of Yaari's representation theorem.

Finally, the third axiom is the crucial one in this framework, as it replaces independence by comonotonic independence. Recall that  $X$  and  $Y$  are comonotonic if  $(X(s) - X(s'))(Y(s) - Y(s')) \geq 0$  for all  $s, s' \in S$ , and the absence of a hedging opportunity between comonotonic prospects justifies the requirement of stability with respect to comonotonic mixtures embodied in this axiom.

We can now state Yaari's representation result.

**Proposition 1** (Yaari). *A preference  $\succsim$  on  $[0, 1]$ -valued prospects satisfies axioms 1-3 if and only if there exists a continuous non-decreasing function  $f$  defined on  $[0, 1]$ , such that  $X \succsim Y$  if and only if  $\gamma(X) \geq \gamma(Y)$ , where  $\gamma$  is defined for all  $X$  as  $\gamma(X) = \int_0^1 f(1 - F_X(t))dt$ .*

This result is interpretable in terms of weighting of outcomes (through the weighting of quantiles). Assume that each of the functions that we consider satisfy the invertibility and regularity conditions needed to perform the following operations. By integration by parts

$$\int_0^1 f(1-F_X(t))dt = \int_0^1 f(1-u)dQ_X(u) = \int_0^1 f(1-u)\frac{d}{du}Q_X(u)du = \int_0^1 f'(1-u)Q_X(u)du.$$

Hence, calling  $\phi(u) = f'(1-u)$ , we have the representation of  $\succsim$  with the functional  $\int_0^1 \phi(t)Q_X(t)dt$ . Hence increasing  $f$  corresponds to positive  $\phi$ , which can be interpreted as a weighting of the quantiles of the prospect  $X$ .

As noted in [34], the functional  $\gamma$  is positively homogeneous and constant additive. Therefore,  $\gamma$  satisfies linearity in payments and has the property that  $\gamma(\gamma(X)) = \gamma(X)$ , so that  $\gamma(X)$  is the certainty equivalent of  $X$  for the decision maker characterized by  $\succsim$ .

**1.2. Risk aversion.** We now turn to the characterization of risk averse decision makers among those satisfying axioms 1-3. We define increasing risk as in Rothschild and Stiglitz ([21]). The formulations in the first part of the definition below are equivalent by the Blackwell-Sherman-Stein theorem (see, for instance, Chapter 7 of [25]).

**Definition 1** (Concave ordering, risk aversion). *a) A prospect  $Y$  is dominated by  $X$  in the concave ordering, denoted  $Y \leq_{cv} X$ , when the equivalent statements (i) or (ii) hold:*

- (i) *for all continuous concave functions,  $\mathbb{E}f(Y) \leq \mathbb{E}f(X)$ , where  $\mathbb{E}$  is the expectation operator.*
- (ii)  *$Y$  has the same distribution as  $\hat{Y}$  where  $(X, \hat{Y})$  is a martingale, i.e.,  $\mathbb{E}(\hat{Y}|X) = X$  ( $\hat{Y}$  is sometimes called a mean preserving spread of  $X$ ).*

*b) The preference relation  $\succsim$  is called risk averse if  $X \succsim Y$  whenever  $X \geq_{cv} Y$ .*

Notice that  $x \rightarrow x$  and  $x \rightarrow -x$  are both continuous and concave function. Therefore, condition (i) in the first part of the definition implies that  $\mathbb{E}[X] = \mathbb{E}[Y]$  is necessary for a concave ordering relationship between  $X$  and  $Y$  to exist. With this definition, we can recall the characterization of risk averse preferences satisfying Axioms 1-3 as those with convex  $f$  (see Section 5 of [34] or Theorem 3.A.7 of [25]).

**Proposition 2.** *A preference relation  $\succsim$  satisfying Axioms 1-3 is risk averse if and only if the function  $f$  in Theorem 1 is convex.*

This monotonicity of the derivative of  $f$  has the natural interpretation that risk averse decision makers evaluate prospects by giving high weights to low quantiles (corresponding to low values of the prospect) and low weights to high quantiles. Indeed, with the formulation  $\gamma(X) = \int_0^1 \phi(u)Q_X(u)du$  and the identification  $\phi(u) = f'(1-u)$ , an increasing convex  $f$  corresponds to positive decreasing  $\phi$ , and therefore to a weighting scheme in which low quantiles (corresponding to unfavorable outcomes) receive high weights and high quantiles (corresponding to favorable outcomes) receive low weights.

## 2. MULTIVARIATE QUANTILES AND COMONOTONICITY

The main ingredients in our multivariate representation theorem are the multivariate extensions of quantiles and comonotonicity. As we shall see, the two are intimately related.

**2.1. Multivariate quantiles.** We first note that the quantile of a random variable can be characterized as an increasing rearrangement of the latter. Hence, by classical rearrangement inequalities, quantiles are solutions to maximum correlation problems. More precisely, by the rearrangement inequality of Hardy, Littlewood and Pólya [12], we have the following well known equality:

$$\int_0^1 Q_X(t)t d(t) = \max \{ \mathbb{E}[XU] : U \text{ uniformly distributed on } [0, 1] \}, \quad (2.1)$$

where the quantile function  $Q_X$  has been defined above. This variational characterization is crucial when generalizing Yaari's representation theorem to the multivariate setting. Indeed, consider now a random vector  $X$  on  $\mathbb{R}^d$  and a reference distribution  $\mu$  on  $\mathbb{R}^d$ , with  $U$  distributed according to  $\mu$ . We introduce *maximum correlation functionals* to generalize the variational formulation of (2.1).

**Definition 2** (Maximal correlation functionals). *A functional  $\varrho_\mu : L_d^2 \rightarrow \mathbb{R}$  is called a maximal correlation functional with respect to a reference distribution  $\mu$  if for all  $X \in L_d^2$ ,*

$$\varrho_\mu(X) := \sup \left\{ \mathbb{E}[X \cdot \tilde{U}] : \tilde{U} =_d \mu \right\}.$$

This functional has an important geometric interpretation as the support function of the equidistribution class of  $\mu$ . It follows from the theory of optimal transportation (see Theorem 2.12(ii), p. 66 of [30]) that if  $\mu$  is absolutely continuous (which will be assumed

throughout the rest of the paper), then there exists a convex lower semi-continuous function  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  and a random vector  $U$  distributed according to  $\mu$  such that  $X = \nabla V(U)$  holds  $\mu$ -almost surely, and such that  $\varrho_\mu(X) = \mathbb{E}[\nabla V(U) \cdot U]$ . In that case, the pair  $(U, X)$  is said to achieve the *optimal quadratic coupling* of  $\mu$  with respect to the distribution of  $X$ . The function  $V$  is called the *Kantorovitch potential* of  $X$  with respect to  $\mu$  or the *transportation potential* from  $\mu$  to the probability distribution of  $X$ <sup>1</sup>. This shows that the gradient of the convex function  $V$  thus obtained satisfies the multivariate analogue of equation (2.1). We therefore adopt  $\nabla V$  as our notion of a generalized quantile.

**Definition 3** ( $\mu$ -quantile). *The  $\mu$ -quantile function of a random vector  $X$  on  $\mathbb{R}^d$  with respect to an absolutely continuous distribution  $\mu$  on  $\mathbb{R}^d$  is defined by  $Q_X = \nabla V$ , where  $V$  is the Kantorovitch potential of  $X$  with respect to  $\mu$ .*

**2.2. Multivariate comonotonicity.** Two univariate prospects  $X$  and  $Y$  are comonotonic if there is a prospect  $U$  and non-decreasing maps  $T_X$  and  $T_Y$  such that  $Y = T_Y(U)$  and  $X = T_X(U)$  almost surely or, equivalently,  $\mathbb{E}[UX] = \max \left\{ \mathbb{E}[X\tilde{U}] : \tilde{U} =_d U \right\}$  and  $\mathbb{E}[UY] = \max \left\{ \mathbb{E}[Y\tilde{U}] : \tilde{U} =_d U \right\}$ . Comonotonicity is hence characterized by maximal correlation between the prospects over the equidistribution class. This variational characterization (where products will be replaced by scalar products) will be the basis for our generalized notion of comonotonicity.

**Definition 4** ( $\mu$ -comonotonicity). *Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  with finite second moments. A collection of random vectors  $X_i \in L_d^2$ ,  $i \in I$ , are called  $\mu$ -comonotonic if one has*

$$\varrho_\mu \left( \sum_{i \in I} X_i \right) = \sum_{i \in I} \varrho_\mu(X_i).$$

When  $\mu$  is absolutely continuous, it follows from the representation of  $\varrho_\mu$  that the family  $X_i$  is  $\mu$ -comonotonic if and only if there exists a vector  $U$  distributed according to  $\mu$  such that  $U \in \operatorname{argmax}_{\tilde{U}} \{ \mathbb{E}[X_i \cdot \tilde{U}] \}$ ,  $\tilde{U} =_d \mu$  for all  $i \in I$ . In other words, the  $X_i$ 's can be rearranged simultaneously so that they achieve maximal correlation with  $U$ . Note that

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<sup>1</sup>Note that  $V$  is convex, hence differentiable except on set of measure zero by Rademacher's Theorem (Theorem 2.4 in [19]), so that the expression  $\mathbb{E}[\nabla V(U) \cdot U]$  above is well defined.

when the distributions of the random vectors are absolutely continuous, then the concept of  $\mu$ -comonotonicity is transitive.

**Proposition 3.** *Suppose that  $X$  and  $Y$  are  $\mu$ -comonotonic, and that  $Y$  and  $Z$  are  $\mu$ -comonotonic, with the distribution of  $Y$  assumed to be absolutely continuous. Then  $X$  and  $Z$  are  $\mu$ -comonotonic.*

*Comonotonic allocations and Pareto efficiency.* It is worth discussing this notion of comonotonicity as it does not seem that obvious that this should be the right generalization of the classical univariate notion of comonotonicity. The main motivation for introducing it is to generalize the equivalence between comonotonic and Pareto efficient allocations, which holds in dimension 1. In dimension one, it is known since a seminal paper of Landsberger and Meilijson [15] that a risk-sharing allocation is Pareto efficient with respect to the concave order if and only if it is comonotonic. That is, given any comonotone allocation, it is not possible to find another allocation such that each risky endowment would be preferred under the new allocation by every risk-averse decision makers over endowments in the previous allocation. Multivariate generalization of this equivalence is not obvious, but it turns out that, as recently shown by Carlier, Dana and Galichon [6], the essence of this result can be extended to the multivariate case, when comonotonicity is replaced by *multivariate comonotonicity*, if one defines an allocation to be comonotonic in the multivariate sense if and only if it is  $\mu$ -comonotonic for some measure  $\mu$  with enough regularity. In our view, this result strongly supports the claim that our notion of comonotonicity is in some sense the “natural” multivariate extension of comonotonicity.

*Interpretation of the reference measure.* We now discuss the behavioral interpretation of  $\mu$ . As we shall see in Theorem 1, the generalization of the Yaari preferences to the multivariate case will lead us to define a utility functional  $\gamma$  over prospects such that  $\gamma(X) = \mathbb{E}[X \cdot \phi(\tilde{U})]$  for some prospect  $\tilde{U}$  which is correlated to  $X$ . The exact construction of  $\tilde{U}$  will be detailed later on, but let us point out at this point that  $\tilde{U}$  is an index such that  $X \cdot \tilde{U}$  measure how favorable the outcome is for the decision maker, which has distribution  $\mu$ , that is it satisfies  $\tilde{U} =_d \mu$ . One sees that  $\phi(\tilde{U})$  is a weighting of the contingent outcome  $X$ , so that  $\gamma$  over- or under-weights prospects in each state using weights  $\phi(\tilde{U})$ . Hence the dispersion of  $\mu$  will induce departure from risk-neutrality. In the limit where  $\mu$  is the distribution of a constant  $u_0$ , one has in that limit  $\gamma(X) = \mathbb{E}[X \cdot \phi(u_0)] = \mathbb{E}[X] \cdot \phi(u_0)$  and one recovers the case of

a risk-neutral decision maker. On the contrary, when  $\mu$  exhibits a lot of dispersion, then the variance of  $\phi(\tilde{U})$  will be high in general, so that the “favorable” outcomes (in the sense that  $X \cdot \tilde{U}$  is high) will be getting less weight, at least if  $\phi(\tilde{U}) = -\alpha\tilde{U}$ . This induces risk aversion. When  $\phi(\tilde{U})$  differs from a rescaling of  $\tilde{U}$ , there may be some discrepancy between the weighting of a given state and how favorable it is, hence the variance of  $\mu$  is no longer directly associated with risk aversion. This will be discussed in Section 4.

It is also worth discussing the equilibrium implications of  $\mu$ . Again, anticipating on Section 4.3,  $\mu$  will be related to the distribution of the state prices in an economy at equilibrium. More precisely,  $\mu$  coincides with the distribution of the stochastic discount factor in an economy at equilibrium. One way to understand this is as follows. According to the result in [6], Pareto efficient allocations coincide (up to technicalities) with  $\mu$ -comonotonic allocations. These efficient allocations can in turn be interpreted as equilibrium allocations of agents with the preferences generalizing Yaari to the multivariate case characterized in Theorem 2. Then the state price supporting this equilibrium should coincide (up to a factor) with the gradient of these utilities. As we shall see, this implies that the stochastic discount factor is distributed according to  $\mu$ . This will be explained in detail in Section 4.3, where the connection is also made with Machina’s local utility functions.

*Relation with other multivariate notions of comonotonicity.* Puccetti and Scarsini [18] have also applied the theory of optimal transportation to generalize the notion of comonotonicity to the multivariate setting. They also review other possible multivariate extension of comonotonicity, including ours. But the concept they favor differs from ours in the sense that according to their notion of multivariate comonotonicity (which they call *c-comonotonicity*), two vectors  $X$  and  $Y$  are *c-comonotonic* if and only if  $X$  and  $Y$  are an optimal quadratic coupling. That is,  $X$  and  $Y$  are *c-comonotonic* if and only if there is a convex function  $V$  such that  $Y = \nabla V(X)$  holds almost surely. However, unlike  $\mu$ -comonotonicity, *c-comonotonicity* is in general not transitive, and does not seem to be related to efficient risk-sharing allocations and equilibrium.

[24] introduces an internal notion of comonotonicity: if a decision maker evaluates prospects according to  $\succsim$ , then Schmeidler-comonotonicity of two prospects  $X$  and  $Y$  means that for all pairs of states of the world  $(s, t)$ ,  $X(s) \succsim X(t)$  implies  $Y(s) \succsim Y(t)$ , i.e. prospects  $X$  and

$Y$  are more desirable in the same states of the world. In contrast, we extend the Weymark-Yaari motivation in our definition of comonotonicity, which can be related to the stochastic discount factor as we have just discussed. The two notions have no obvious relation, as we see by considering two  $\mu$ -comonotonic prospects  $X$  and  $Y$  and imposing Schmeidler comonotonicity. By  $\mu$ -comonotonicity, there exists  $U =_d \mu$  and generalized quantile functions  $Q_X$  and  $Q_Y$  such that  $X = Q_X(U)$  and  $Y = Q_Y(U)$ . Schmeidler-comonotonicity of  $X$  and  $Y$  would require that the univariate random variables  $(\mathbb{E}U) \cdot X$  and  $(\mathbb{E}U) \cdot Y$  are comonotonic in the usual sense. Although equivalent in dimension one, none of these concepts implies the other one in higher dimension.

### 3. MULTIVARIATE REPRESENTATION THEOREM

Now that we have given a formalization of the notion of maximal correlation in a law invariant sense that is suitable for a multivariate extension of Yaari's dual theory, we can proceed to generalize Yaari's representation result to the case of a preference relation among multivariate prospects. We consider prospects, which are elements of  $L_d^2$ .

**Axiom 1'**. The preference  $\succsim$  is represented by a continuous functional  $\gamma$  on  $L_d^2$ , such that at at least one point its Fréchet derivative exists and is non-zero.

Note first that although Axiom 1' is stated directly in terms of the representing functional for simplicity of exposition, it is a rather mild requirement. As in [7], the functional will not be Fréchet differentiable in all points, but we only require differentiability in one point.

Given sufficient regularity, first order stochastic dominance can be characterized equivalently by pointwise dominance of cumulative distribution functions or pointwise dominance of quantile functions. It is the latter that we adopt for our multivariate definition.

**Definition 5** ( $\mu$ -first order stochastic dominance). *A prospect  $X$   $\mu$ -first order stochastically dominates prospect  $Y$  relative to the componentwise partial order  $\geq$  on  $\mathbb{R}^d$  if  $Q_X(t) \geq Q_Y(t)$  for almost all  $t \in \mathbb{R}^d$ , where  $Q_X$  and  $Q_Y$  are the generalized quantiles of  $X$  and  $Y$  with respect to a distribution  $\mu$  on  $\mathbb{R}^d$ .*

Note that for any  $U =_d \mu$ , we have  $Q_X(U) =_d X$  and  $Q_Y(U) =_d Y$ . Then if  $X$   $\mu$ -first order stochastically dominates  $Y$  then  $Q_X(U) \geq Q_Y(U)$  almost surely. Hence,  $\hat{X} \geq \hat{Y}$  almost

surely for some  $\hat{X} =_d X$  and  $\hat{Y} =_d Y$ , which is the “usual multivariate stochastic order” (see [25] p. 266). The converse does not hold in general.

The remaining two axioms require fixing an absolutely continuous reference probability distribution  $\mu$  on  $\mathbb{R}^d$ .

**Axiom 2’.** The preference  $\succsim$  preserves  $\mu$ -first order stochastic dominance in the sense that if prospect  $X$   $\mu$ -first order stochastically dominates prospect  $Y$ , then  $X \succsim Y$ , and if  $X$   $\mu$ -first order strictly stochastically dominates prospect  $Y$ , then  $X \succ Y$ .

**Axiom 3’.** If  $X, Y, Z$  are  $\mu$ -comonotonic prospects, then for any  $\alpha \in [0, 1]$ ,  $X \succsim Y$  implies  $\alpha X + (1 - \alpha)Z \succsim \alpha Y + (1 - \alpha)Z$ .

This extension of the comonotonicity axiom is the key to the generalization of the dual theory to multivariate prospects. The statement of Axiom 3 is unchanged, but the notion of comonotonicity is now dependent on a reference distribution  $\mu$ . The prospects  $X, Y, Z$  are comonotonic, or more precisely  $\mu$ -comonotonic, if they are all *maximally correlated* in the law invariant sense of (4) with a reference  $U$  (where  $U$  has distribution  $\mu$ ).

We are now in a position to state the multivariate extension of Yaari’s representation.

**Theorem 1** (Multivariate Representation). *A preference relation on square integrable multivariate prospects satisfies Axioms 1’, 2’ and 3’ relative to a reference probability measure  $\mu$  if and only if there exists a function  $\phi$  such that for  $U =_d \mu$ ,  $\phi(U)$  is square integrable,  $\phi(U) \in (\mathbb{R}_-)^d$  almost surely and such that for all pairs  $X, Y$ ,  $X \succsim Y$  if and only if  $\gamma(X) \geq \gamma(Y)$ , where  $\gamma$  is defined for all  $X$  by  $\gamma(X) = \mathbb{E}[Q_X(U) \cdot \phi(U)]$ , where  $Q_X$  is the  $\mu$ -quantile of  $X$ .*

When  $d = 1$ , the representation is independent of  $\mu$  and we recover the result of Proposition 1. As in the univariate case, the decision maker assesses prospects with a weighting scheme  $\phi$  of quantiles of this risk. Because  $\gamma$  in Theorem 1 is positively homogeneous,  $\gamma(X)$  is the certainty equivalent of  $X$ , and because  $\gamma$  remains constant additive,  $\succsim$  satisfies *linearity in payments*, i.e., for any positive real number  $a$  and any  $b \in \mathbb{R}^d$  (identified with a constant multivariate prospect),  $\gamma(aX + b) = a\gamma(X) + b$ .

## 4. RISK AVERSION, LOCAL UTILITY FUNCTION

In this section, we consider the question of representing those decision makers satisfying Axioms 1', 2' and 3' that are risk averse in the sense of Definition 1. We then show that the *local utility function* in the sense of [17] is easily computable and provides an interpretation of the reference distribution  $\mu$ .

**4.1. Risk aversion.** For our characterization of risk averse Yaari decision makers, we need to generalize the notion of mean preserving spread to the multivariate setting.

**Proposition 4** (Concave ordering). *For any prospects  $X$  and  $Y$  whose respective distributions are absolutely continuous, the following properties are equivalent.*

- (a) For every bounded concave function  $f$  on  $\mathbb{R}^d$ ,  $\mathbb{E}f(X) \geq \mathbb{E}f(Y)$
- (b)  $Y =_d \hat{Y}$ , with  $\mathbb{E}[\hat{Y}|X] = X$ .
- (c)  $\varrho_\mu(X) \leq \varrho_\mu(Y)$  for every probability measure  $\mu$ .
- (d)  $X$  belongs to the closure of the convex hull of the equidistribution class of  $Y$ .
- (e) For every upper semi-continuous law-invariant concave functional  $\Phi : L_d^2 \rightarrow \mathbb{R}$ , one has  $\Phi(X) \geq \Phi(Y)$ .

When any of the properties above hold, one says that  $Y$  is dominated by  $X$  in the concave ordering, denoted  $Y \leq_{cv} X$ .

Statements (a) and (b) are identical in the multivariate case as in [21]. The equivalence between the two is a classical result that can be traced back at least to [27] (see section A.2 for details). The interpretation of the ordering as a preference ordering for all risk aversion expected utility maximizers (a) and as an ordering of mean-preserving spreads (b) also carry over to the multivariate case. Statement (d) is the continuous equivalent to multiplication by a bi-stochastic matrix, which characterizes uniform majorization in the multi-attribute inequality literature (see [14]). Statement (d) of this theorem provides an important geometric interpretation of the ordering.

As in the univariate case and definition 1(b), risk averse decision makers will be defined by aversion to mean-preserving spreads. It turns out that imposing risk aversion on a preference relation that satisfies axioms 1', 2' and 3' is equivalent to requiring the following property, sometimes called *preference for diversification*.

**Axiom 4.** For any two preference equivalent prospects  $X$  and  $Y$  (i.e., such that  $X \sim Y$ ), convex combinations are preferred to either of the prospects, (i.e., for any  $\alpha \in [0, 1]$ ,  $\alpha X + (1 - \alpha)Y \succeq X$ ).

This is formalized in the following theorem, which gives a representation for risk averse Yaari decision makers.

**Theorem 2.** In dimension  $d \geq 2$ , for a preference relation satisfying Axioms 1', 2' and 3', the following statements are equivalent:

- (a)  $\succeq$  is risk averse, namely  $X \succeq Y$  whenever  $X \geq_{cv} Y$ .
- (b)  $\succeq$  satisfies Axiom 4.
- (c) The function  $\phi$  involved in the representation of the preference relation in Theorem 1 satisfies  $-\phi(u) = \alpha u + u_0$  for  $\alpha > 0$  and  $u_0 \in \mathbb{R}^d$ .

So, in the multivariate setting the functional  $\gamma$  is convex if and only if  $\phi(x) = -\alpha x - x_0$  for some  $\alpha$  real positive and  $x_0 \in \mathbb{R}^d$ . This is a major difference with dimension one, where the functional is convex if and only if  $-\phi$  is a non-decreasing map. This implies that a multivariate Yaari risk averse decision maker is entirely characterized by the reference distribution  $\mu$ .

**4.2. Local Utility Function.** Throughout the rest of the paper, we shall assume the conditions in Theorem 2 are met. By law-invariance, we denote  $\gamma(P) := \gamma(X)$ , where  $X =_d P$ . Without loss of generality, we shall also assume that  $\phi(u) = -u$ , thus  $\gamma(P) := -\mathbb{E}[\nabla V_P(U) \cdot U]$ , where  $V_P = V_X$  is the Kantorovich transportation potential from the reference probability distribution  $\mu$  of  $U$  to the probability distribution  $P$  of  $X$ . As we have seen, the gradient  $\nabla V_P$  of this transportation potential is the  $\mu$ -quantile function of distribution  $P$ . As shown in [17], when smoothness requirements are met, a local analysis can be carried out in which a (risk-averse) non-Expected Utility function behaves for small perturbations around a fixed risk in that same way as a (concave) utility function. Formally, the local utility function is defined as  $u(x|P) = D_P \gamma(x)$ , where  $D_P \gamma$  is the Fréchet derivative of  $\gamma$  at  $P$ . Denoting by  $V^*(x) = \sup_u [u \cdot x - V(u)]$  the *Legendre-Fenchel transform* of a

convex lower semicontinuous function  $V$ , we have:

$$\begin{aligned}\gamma(P) &= \mathbb{E}[\nabla V_P(U) \cdot U] = \max \{ \mathbb{E}[X \cdot U] : X =_d P, U =_d \mu \} \\ &= \min \left\{ \int V(u) d\mu(u) + \int V^*(x) dP(x) : V \text{ convex and l.s.c.} \right\} \\ &= \int V_P(u) d\mu(u) + \int V_P^*(x) dP(x)\end{aligned}$$

by the duality of optimal transportation (see, for instance, Theorem 2.9, p. 60 of [30]).

Defining  $f(V, Q) = \int V d\mu + \int V^* dQ$ , we have  $\gamma(P) = -\inf_V f(V, P)$ . Hence, an envelope theorem argument formally yields  $u(x|P) = D_P \gamma(x) = -V_P^*(x)$ . Therefore, *the local utility function is  $-V_P^*$ , the (negative of the) Legendre-Fenchel transform of the Kantorovich transportation potential  $V_P$* . This point sheds light on the economic interpretation of this potential, thanks to Machina's theory of local utility. The function  $-V_P^*$  is concave, which is consistent with the risk aversion of a Yaari decision maker given the assumptions of theorem 2. Note that for univariate prospects,  $u(x|P) = -V_P^*(x) = \int_{-\infty}^x F_X(z) dz$ , so that we recover the fact that when  $X =_d P$  is a mean preserving spread of  $Y =_d Q$ ,  $u(z|P) \leq u(z|Q)$  for all  $z$ .

**4.3. Economic interpretation of  $\mu$ .** The previous result will allow us to complete the discussion of the economic interpretation of the reference distribution  $\mu$ . Consider an economy where one of the agents (whom we shall refer to as ‘‘Yaari’’) has preferences as in Theorem 2, with reference distribution  $\mu$ . Assume that there is a risk sharing equilibrium in this economy, which is supported by stochastic discount factor  $\xi$ . Hence, if the original risky endowment of the agent is  $X_0$ , the agent's budget set is  $\{X : \mathbb{E}[(X - X_0) \cdot \xi] = 0\}$ . Then the demand for risk  $X$  of Yaari is therefore  $\max_{\hat{X}} \gamma(\hat{X})$  subject to  $\mathbb{E}[(\hat{X} - X_0) \cdot \xi] = 0$ . But because Yaari has been assumed to be risk-averse,  $\gamma$  is concave, and the demand for risk  $X$  satisfies the local optimality condition  $\max_{\hat{X}} \mathbb{E} \left[ u(\hat{X}|P) \right]$  subject to  $\mathbb{E}[(\hat{X} - X_0) \cdot \xi] = 0$ . and first order conditions yield  $\nabla u(X|P) = \lambda \xi$  where  $\lambda$  is the Lagrange multiplier associated to the budget constraint,  $\lambda \neq 0$  unless the equilibrium is No-Trade. But as explained above,  $u(x|P) = -V_P^*(x)$ , hence  $\nabla u(X|P) = -\nabla V_P^*(X)$ . Now, by definition of the Kantorovich potential  $V_P$  from  $\mu$  to  $P$ ,  $\nabla V_P^*(X) =_d \mu$ . Hence  $-\lambda \xi =_d \mu$  which implies that  $\mu$  is (up to scale) the distribution of the stochastic discount factor  $\xi$ . Hence, when there is a Yaari decision maker with reference scenario  $\mu$  in the economy, the stochastic discount factor should be distributed according to  $\mu$ . This result is an extension of the well-known result

that states that when there is a risk-neutral decision maker in the economy, the stochastic discount factor should equal one, that is, the risk-neutral probability should coincide with the actuarial probability. To summarize, if a risk-sharing equilibrium is to exist with a Yaari risk-averse decision maker with reference scenario  $\mu$ , then  $\mu$  should coincide with the distribution of the stochastic discount factor. Therefore  $\mu$  is related to the distribution of the state prices.

## 5. RELATION WITH MULTI-ATTRIBUTE INEQUALITY MEASUREMENT

The theory developed here has implications for inequality rankings of allocations of multiple attributes (such as income, education, environmental quality etc...) in a population. [2] recognized the relevance of stochastic orderings to the measurement of inequality and its foundation on principles such as the desirability of Pigou-Dalton transfers (also known as Pigou-Dalton Majorization). [31] added to Pigou-Dalton Majorization a principle of comonotonic independence, which he interpreted as neutrality to the source of variation in income, and obtained a class of social evaluation functions, which he called generalized Gini evaluation functions. The functional form is identical to the decision functional derived independently on a continuous state space by [34]. Indeed, [14] notes the formal equivalence between the problem of decision under risk and the measurement of inequality. The random vector of risks, or prospects that we consider in the present work can be interpreted as an allocation of multiple attributes over a continuum of individuals. With this interpretation, states of the world are identified with individuals in the population and the decision function  $\gamma$  is interpreted as a social evaluation function. Law invariance (Yaari neutrality, i.e. insensitivity to relabellings of the states of the world) of the decision functional is thus equivalent to anonymity of the social evaluation function. The ranking of ordinally equivalent allocations obtained through Pigou-Dalton majorization (see[14]) corresponds to the concave ordering discussed in proposition 4. More precisely, the mean-preserving spread characterization (b) in proposition 4 is equivalent to (d), which is the infinite-dimensional analogue of multiplication by a bi-stochastic matrix. Hence, our risk averse multivariate Yaari decision functional can be interpreted as a social evaluation function for allocations of multiple attributes, which satisfies anonymity, monotonicity and Pigou-Dalton majorization in the sense of theorem 3 in [14].

The literature achieves functional forms for social evaluation functions in the multiattribute case by adding two distinct types of majorization principles that allow the comparison of non-ordinally equivalent social evaluations. [28] and [29] consider correlation increasing transfers. [11] extend generalized Gini social evaluation functions to the multivariate case with a comonotonic independence axiom. Two allocations are said to be comonotonic if all individuals are ranked identically in all attributes (i.e. the richest is also the most educated etc...), and the ranking between two comonotonic allocations is not reversed by the addition of a comonotonic allocation. They use an attribute separability axiom (Axiom A in [33]) to reduce the dimensionality of the problem via independence of the attributes. Specifically, Theorem 4 of [11] is a special case of our theorem 2 when the attribute vector  $X$  and the reference distribution  $\mu$  both have independent marginals. Our representation can also incorporate trade-offs between attributes and attitude to correlations between attributes of the kind that are entertained in [29], but not restricted to the latter. Correlation aversion would correspond to perceived substitutability, but perceived complementarity can also be entertained in our approach, thereby circumventing Bourguignon's critique of correlation increasing majorization (in [4]).

## 6. CONCLUSION

We have developed notions of quantiles and comonotonicity for multivariate prospects, thus allowing for the consideration of choice among vectors of payments in different currencies, at different times, in different categories of goods, etc. The multivariate concepts of quantiles and comonotonicity were used to generalize Yaari's dual theory of choice under risk, where decision makers that are insensitive to hedging of comonotonic risks are shown to evaluate prospects using a weighted sum of quantiles. Risk averse decision makers were shown to be characterized within this framework by a reference distribution, making the dual theory as readily applicable as expected utility. Risk attitude is also analyzed from the point of view of the local utility function and implications for the ranking of increasing risk aversion is the topic of further work. Applications of the representation theorem to the measurement of multi-attribute inequality are also discussed and the flexibility of the decision function in its handling of attitude to correlation between attributes is a promising feature.

## APPENDIX A. PROOF OF RESULTS IN THE MAIN TEXT

**A.1. Proof of Proposition 3.** By definition, there are two convex lower semi-continuous functions  $V_1$  and  $V_2$  and a random vector  $U =_d \mu$  such that  $X = \nabla V_1(U)$  and  $Y = \nabla V_2(U)$  almost surely. Similarly, there are convex functions  $V_3$  and  $V_4$  and a random vector  $\tilde{U}$  such that  $Y = \nabla V_3(\tilde{U})$  and  $Z = \nabla V_4(\tilde{U})$ . Now the assumptions on the absolute continuity of  $\mu$  and the distribution of  $Y$  imply that  $\nabla V_2$  is essentially unique. Hence,  $\nabla V_2 = \nabla V_3$ , and, therefore,  $U = \tilde{U}$  holds almost surely. It follows that  $X$  and  $Z$  are  $\mu$ -comonotonic.  $\square$

**A.2. Proof of Proposition 4.** The equivalence between (a) and (b) is a famous result stated and extended by many authors, notably Hardy, Littlewood, Polya, Blackwell, Stein, Sherman, Cartier, Fell, Meyer and Strassen. See Theorem 2 of [27] for an elegant proof. We now show that (b) implies (c). Suppose (b) holds. As explained in Section 2.1, there exists a map  $\zeta$  such that  $\varrho_\mu(X) = \mathbb{E}[\zeta(X) \cdot X]$  and  $\zeta(X) =_d \mu$ . Now,  $\mathbb{E}[\zeta(X) \cdot X] = \mathbb{E}[\zeta(X) \cdot \mathbb{E}[\hat{Y}|X]] = \mathbb{E}[\zeta(X) \cdot \hat{Y}]$ , which is less than  $\varrho_\mu(Y)$ . (c) implies (d): Indeed, the convex closure  $\overline{\text{co}}(\text{equi}(Y))$  of the equidistribution class of  $Y$  is a closed convex set and hence characterized by its support functional  $\varrho_\mu(Y)$ . Therefore,  $X \in \overline{\text{co}}(\text{equi}(Y))$  is equivalent to  $\mathbb{E}[Z \cdot X] \leq \varrho_\mu(Y)$  for all  $Z$ , which in turn is equivalent to  $\varrho_\mu(X) \leq \varrho_\mu(Y)$ . (d) implies (e): Indeed, if  $X \in \overline{\text{co}}(\text{equi}(Y))$ , then there is a sequence  $(Y_k^n)_{k \leq n}$  of random vectors each distributed as  $Y$  and positive weights  $\alpha_k^n$  such  $\sum_{k=1}^n \alpha_k^n = 1$ , and  $X = \lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_k^n Y_k^n$ . Then, for any law invariant concave functional, we have  $\Phi(\sum_{k=1}^n \alpha_k^n Y_k^n) \leq \sum_{k=1}^n \alpha_k^n \Phi(Y_k^n) = \Phi(Y)$  and the conclusion follows by upper-semicontinuity. Finally, (e) implies (a) because when  $\mathcal{L}_X$  is absolutely continuous, for any bounded concave function  $f$ ,  $X \mapsto \mathbb{E}f(X)$  is a law invariant concave upper semi-continuous functional.  $\square$

**A.3. Proof of Theorem 1.** Note first that  $\gamma$  defined for all prospect  $X$  by  $\gamma(X) = \mathbb{E}[Q_X(U) \cdot \phi(U)]$ , for a function  $\phi$  such that  $\phi(U) \in (\mathbb{R}_-)^d$  is Lipschitz and monotonic, so that Axioms 1' and 2' are satisfied for a preference relation represented by  $\gamma$ . Finally, comonotonic independence follows directly from the fact that for any two prospects  $X$  and  $Y$ , the generalized quantile functions  $Q_X$ ,  $Q_Y$  and  $Q_{X+Y}$  satisfy  $Q_{X+Y}(U) = Q_X(U) + Q_Y(U)$ . We now show this fact. By the definition of the generalized quantile functions, we have  $\mathbb{E}[Q_{X+Y}(U) \cdot U] = \sup_{\tilde{U} =_d U} \mathbb{E}[(X+Y) \cdot \tilde{U}] \leq \sup_{\tilde{U} =_d U} \mathbb{E}[X \cdot \tilde{U}] + \sup_{\tilde{U} =_d U} \mathbb{E}[Y \cdot \tilde{U}] = \mathbb{E}[Q_X(U) \cdot U] + \mathbb{E}[Q_Y(U) \cdot U]$ . On the other hand, we also have  $\mathbb{E}[Q_{X+Y}(U) \cdot U] = \sup_{\tilde{Z} =_d X+Y} \mathbb{E}[\tilde{Z} \cdot U] \geq \mathbb{E}[(Q_X(U) + Q_Y(U)) \cdot U]$ , since by construction,  $Q_X(U) =_d X$  and  $Q_Y(U) =_d Y$ , and the desired equality follows. Conversely, we now prove that a preference relation  $\succsim$  satisfying Axioms 1', 2' and 3' is represented by a functional  $\gamma$  defined for all prospect  $X$  by  $\gamma(X) = \mathbb{E}[Q_X(U) \cdot \phi(U)]$ , for a function  $\phi$  such that  $\phi(U) \in (\mathbb{R}_-)^d$  almost surely. By Axiom 1', there exists a functional  $\gamma$  representing  $\succsim$  and there is a point  $Z \in L_d^2$ , where  $\gamma$  is Fréchet differentiable with non-zero gradient  $D$ . Let  $Q_Z$  be the generalized quantile of  $Z$  relative to  $\mu$ . There exists a  $U \in L_d^2$  with distribution  $\mu$  such that  $Z = Q_Z(U)$  almost surely. Let  $X$  and  $Y$

be two prospects in  $L_d^d$  with  $\mu$ -quantile functions  $Q_X$  and  $Q_Y$  respectively. By the definition of  $\mu$ -comonotonicity,  $Q_X(U)$ ,  $Q_Y(U)$  and  $Z = Q_Z(U)$  are  $\mu$ -comonotonic. By Axiom 2',  $\gamma$  is law invariant, so that  $\gamma(X) \geq \gamma(Y)$  is equivalent to  $\gamma(Q_X(U)) \geq \gamma(Q_Y(U))$ . Hence, by Axiom 3',  $\gamma(X) \geq \gamma(Y)$  implies that for any  $0 < \epsilon \leq 1$ , we have  $\gamma(\epsilon Q_X(U) + (1 - \epsilon)Z) \geq \gamma(\epsilon Q_Y(U) + (1 - \epsilon)Z)$ , hence  $\gamma(Z + \epsilon(Q_X(U) - Z)) \geq \gamma(Z + \epsilon(Q_Y(U) - Z))$  and, therefore,  $\gamma(Z) + \mathbb{E}[D \cdot \epsilon(Q_X(U) - Z)] \geq \gamma(Z) + \mathbb{E}[D \cdot \epsilon(Q_Y(U) - Z)] - o(\epsilon)$ , or, finally,  $\mathbb{E}[D \cdot Q_X(U)] \geq \mathbb{E}[D \cdot Q_Y(U)]$ . Suppose now that  $X$  and  $Y$  are two prospects such that  $\mathbb{E}[D \cdot Q_X(U)] = \mathbb{E}[D \cdot Q_Y(U)]$ . We shall show that  $\gamma(Q_X(U)) = \gamma(Q_Y(U))$ , and, hence, that  $\gamma(X) = \gamma(Y)$ , thereby concluding that the functional  $X \mapsto \mathbb{E}[D \cdot Q_X(U)]$  represents  $\succsim$ . Indeed, suppose that  $\mathbb{E}[D \cdot Q_X(U)] = \mathbb{E}[D \cdot Q_Y(U)]$ . We will show shortly that there exists a function  $\phi$  such that  $\mathbb{E}[D \cdot \nabla \phi(U)] > 0$  and, hence, that  $\mathbb{E}[D \cdot (Q_X(U) + \epsilon \nabla \phi(U))] > \mathbb{E}[D \cdot Q_Y(U)]$  and  $\mathbb{E}[D \cdot (Q_X(U) - \epsilon \nabla \phi(U))] < \mathbb{E}[D \cdot Q_Y(U)]$  for any  $\epsilon > 0$ . Using the result above yields  $\gamma(Q_X(U) + \epsilon \nabla \phi(U)) \geq \gamma(Q_Y(U))$  and  $\gamma(Q_X(U) - \epsilon \nabla \phi(U)) \leq \gamma(Q_Y(U))$ . Hence,  $\gamma(Q_X(U)) = \gamma(Q_Y(U))$  by the continuity of  $\gamma$ . Let us now show that  $\mathbb{E}[D \cdot \nabla \phi(U)] = 0$  for all gradient functions  $\nabla \phi$  yields a contradiction. Calling  $V_Z$  the convex function such that  $Z = Q_Z(U) = \nabla V_Z(U)$  almost surely,  $D$  is the Fréchet derivative of  $\gamma$  at  $Z = \nabla V_Z(U)$ . Hence,  $\mathbb{E}[D \cdot \nabla \phi(U)] = 0$  implies that  $\gamma(\nabla(V_Z(U) + \epsilon \phi(U))) = \gamma(\nabla V_Z(U)) + o(\epsilon)$ . This is true for all gradient functions  $\nabla \phi$  and in particular, for  $\phi = (V_\epsilon - V_Z)/\epsilon$ , where  $V_\epsilon$  is such that  $Z_\epsilon = \nabla V_\epsilon(U)$  converges to  $Z$  in  $L^2$ . We then have  $\gamma(Z_\epsilon) - \gamma(Z) = o(\epsilon)$  and, hence,  $D = 0$ , which contradicts Axiom 1'. We have shown that  $\succsim$  is represented by the functional  $X \mapsto \mathbb{E}[D \cdot Q_X(U)]$ . As  $\mu$  is absolutely continuous,  $D$  can be written as  $\phi(U)$  for some function  $\phi$  which takes values in  $(\mathbb{R}_-)^d$  by Axiom 2'.  $\square$

**A.4. Proof of Theorem 2.** (a) implies (b) follows from proposition 4. (b) implies (c): Axiom 4 implies that  $\gamma(Q_X + \alpha(\tilde{X} - Q_X)) \geq \gamma(X)$  for all  $\alpha \in (0, 1]$  and all  $\tilde{X}$  in the equidistribution class of  $X$ . The representation of Theorem 1 implies the differentiability of  $\gamma$  at  $Q_X(U)$  for any  $X$ , call  $D_X$  its gradient. This implies that  $\mathbb{E}[\tilde{X} \cdot D_X] \leq \mathbb{E}[Q_X(U) \cdot D_X]$  for all  $\tilde{X} \stackrel{d}{=} X$ . Hence,  $\gamma(X) = -\varrho_{\mathcal{L}_{\phi(U)}}(X)$ . Thus, by Axiom 3', comonotonicity with respect to  $\mu$  implies comonotonicity with respect to  $\mathcal{L}_{\phi(U)}$ . By Lemma 10 in [9], this implies that  $\phi(U) = -\alpha U - x_0$  for some  $\alpha > 0$  and  $x_0 \in \mathbb{R}^d$ , and the result follows. Finally, we show that (c) implies (a). Assume (c), in which case for all  $X \in L_d^2$ ,  $-\gamma(X) = \alpha \mathbb{E}[Q_X(U) \cdot U] + u_0 \cdot \mathbb{E}[X]$ . Thus,  $-\gamma(X) = \alpha \varrho_\mu(X) + u_0 \cdot \mathbb{E}[X]$ . Therefore, by Proposition 4,  $X \succeq_{cv} Y$  implies  $\varrho_\mu(X) \leq \varrho_\mu(Y)$ , and so  $\gamma(X) \geq \gamma(Y)$ .  $\square$

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