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CONGESTION IN A CITY WITH A CENTRAL BOTTLENECK

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Congestion in a city with a central bottleneck*

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Abstract

We consider dynamic congestion in an urban setting where trip origins are spatially distributed. All travelers must pass through a downtown bottleneck in order to reach their destination in the CBD. Each traveler chooses departure time to maximize general concave scheduling utility. At equilibrium, travelers sort according to their distance to the destination. We construct a welfare maximizing tolling regime, which eliminates congestion. All travelers located beyond a critical distance from the CBD gain from tolling, even when toll revenues are not redistributed, while nearby travelers lose. We discuss our results in the context of acceptability of tolling policies.

Key words: dynamic model; toll policy; spatial differentiation; acceptability

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1 Introduction

This paper presents a model that integrates two prominent features of urban congestion, focusing on the exemplary case of the morning commute. The first feature is that congestion is a dynamic phenomenon in the sense that congestion at one time of day affects conditions later in the day through the persistence of queues. The second feature is that trip origins are spatially distributed. We analyse how these features interact in a city with a central bottleneck and provide results concerning optimal pricing.

The dynamics of congestion were analysed in the seminal Vickrey (1969) bottleneck model (see Arnott et al. (1993)). The bottleneck congestion technology captures the essence of congestion dynamics in a simple and tractable way. Travellers are viewed as having scheduling preferences concerning the timing of trips that have to pass the bottleneck. The analysis concerns Nash equilibrium in the traveller choice of departure time.

The Vickrey (1969) analysis of congestion, however, essentially ignores space. Using the notation of the current paper, travellers are depicted as travelling some distance c (measured in time units) until they reach a bottleneck at time a. They exit the bottleneck to arrive at the destination at time t. They have scheduling preferences, always preferring to depart later and always preferring to arrive earlier. The Vickrey (1969) scheduling preferences can be expressed by the scheduling utility $\alpha \cdot c + \alpha \cdot (t - a) + D(t)$, where α is the value of travel time, $t - a$ is the time spent in the bottleneck and $D(t) = \beta \cdot \max(0, t^* - t) + \gamma \cdot \max(0, t - t^*)$ is a function capturing the cost of being early or late relative to some preferred arrival time t^*. The Vickrey formulation of scheduling preferences is additively separable in trip duration and arrival time and it is linear in trip duration. So it is clear that the distance c to the bottleneck does not matter for the Vickrey analysis of how travellers time their arrival at the bottleneck and the ensuing congestion.[1]

It is not generally true that the distance from trip origins to the destination is irrelevant for the timing of trips. Consider a traveller who always prefer to depart later and always prefer to arrive earlier. Faced by a fixed trip duration that is independent of the departure time, such a traveller will optimally time his trip such that his marginal utility of being at the origin at the departure time equals his marginal utility of being at the destination at the arrival time. If the marginal utility at the origin is decreasing and his marginal utility of being at the destination is increasing, then an increase in trip duration will cause him to depart earlier and

[1] The analysis of the bottleneck model has been developed and extended in various directions by Arnott, de Palma and Lindsey in a series of papers; notably Arnott et al. (1993). These authors use the above $\alpha - \beta - \gamma$ preferences or a version where the function $D(t)$ has a more general form. They always maintain linearity and additive separability of travel time and are hence unable to analyse the consequences of distance for congestion.

to arrive later. In this way the distance can matter for the timing of trips. In this paper we consider travellers with such scheduling preferences.

Congestion can arise when there is a bottleneck and many individuals who want to pass the bottleneck at the same time. It is not a sufficient condition for congestion to arise that travellers have similar scheduling preferences. Trip origins must also be located with similar distances to the bottleneck. If trip origins are sufficiently dispersed, then congestion does not arise as there is no overlap in the times when travellers want to pass the bottleneck. Hence it is clear that the spatial distribution of travel demand is a fundamental determinant of urban congestion.

This paper is the first to allow for spatial heterogeneity in the bottleneck model in a meaningful way. Section 2 presents the model setup. The analysis of Nash equilibrium in section 3 shows that Nash equilibrium exists uniquely. Travellers in Nash equilibrium sort according to their distance to the bottleneck such that those who are closest to the bottleneck reach the destination first. However, in general there is not a monotonous relationship between distance and departure time. It is not necessarily the case that those who are located further away will depart earlier. Section 4 then concerns socially optimal tolling at the bottleneck. The toll can be taken to be zero for the first and last travellers and strictly positive for everybody else. The optimal toll exactly removes queueing. The sequence of arrivals at the destination is preserved from the Nash equilibrium. However, in contrast to the Vickrey analysis with homogenous travellers, arrivals at the destination occur earlier in social optimum than in Nash equilibrium. When the use of toll revenues does not affect the utility of travellers, then the toll just represents a loss for them. This is compensated to some extent by a gain in scheduling utility. Comparing social optimum to Nash equilibrium reveals that those who are located furthest away from the bottleneck will experience a net gain, while those who are located near the bottleneck will experience a net loss. Section 5 illustrates the model numerically. Section 6 concludes. Proofs of lemma and theorems are deferred to the appendix.

2 A spatial model

Consider a city in which a continuum of N identical individuals are spatially distributed. They make one trip each and must all pass through a downtown bottleneck. The trip duration of an individual is the sum of the distance, measured in time units, to the bottleneck and the time spent in the bottleneck.

The bottleneck has a capacity of ψ persons per time unit. The distribution of travel distances c has cumulative distribution F with density f and support $C = [c_0, c_1]$. The situation is illustrated in figure 1.

Let a be the arrival time at the bottleneck and c be the travel distance to the

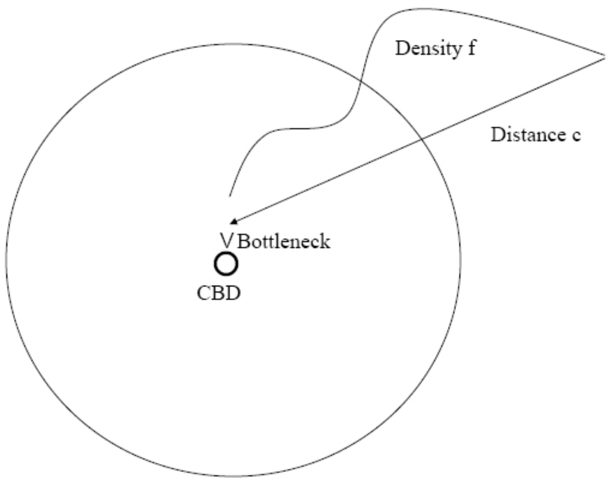


Figure 1: Spatial

bottleneck. Then $d = a - c$ is the departure time from home. Individuals arrive at the bottleneck during some interval $[a_0, a_1]$ at the time varying rate $\rho(a)$. The cumulative arrival rate is $R(a) = \int_{a_0}^{a} \rho(s)\, ds$. In the case when there is queue from time a_0 to time a, then an individual who arrives at entrance of the bottleneck at time a will arrive at the destination at time $R(a)/\psi + a_0$ (Arnott et al. (1993)).

Individuals are identical with preferences concerning the timing and cost of the trip expressed by the utility $u(d, t) - \tau$, where d is the departure time, $t > d$ is the arrival time and τ is the toll payment associated with the trip. Toll revenues are not returned to travellers and their utility is not affected by the use of revenues. This assumption puts a focus on the direct impact of tolling on travellers. All other monetary trip costs are held constant and hence ignored. In brief we shall refer to $u(d, t)$ as the scheduling utility. We shall define a social welfare function to be sum of the individual utilities plus the toll revenue. Hence the welfare function reduces to the sum of the individual scheduling utilities.

Partial derivatives of $u(\cdot, \cdot)$ are assumed to exist up to second order and are denoted as u_1, u_{12} etc. We assume that $u(\cdot, \cdot)$ is strictly concave and that $u_1 > 0 > u_2$. In addition, we shall refer to the following conditions regarding the derivatives of u.

Condition 1 $\forall d \leq t : u_{11}(d, t) - \frac{u_1(d,t)}{u_2(d,t)} u_{12}(d, t) < 0$

Condition 2 $\forall d \leq t : u_{12}(d,t) - \frac{u_1(d,t)}{u_2(d,t)} u_{22}(d,t) < 0$

Condition 3 $\forall d \leq t : u_{11}(d,t) + u_{12}(d,t) < 0$

Condition 4 $\forall d \leq t : u_{12}(d,t) + u_{22}(d,t) < 0$

Conditions (1) and (2) are used to prove that Nash equilibrium is unique. Together they ensure that $u_1(a-c,a)/u_2(a-c,a)$ is decreasing as a function of a. This is a special case of the Spence-Mirrlees condition. Conditions (3) and (4) ensure that $u_1(a-c,a)$ and $u_2(a-c,a)$ are decreasing as functions of a. We assume that $u(a-c,a)$ attains a maximum as a function of a for any $c \geq 0$. This simply assures that travellers have an optimal time of departure when trip duration is constant. We also assume that $u_1(a-c,a) + u_2(a-c,a)$ ranges over all of $\mathbb{R}$ as a varies.

As noted above, the key to the analysis is the formulation of scheduling preferences that are not additively separable and linear in trip duration. Vickrey (1973) did in fact describe such scheduling preferences. Tseng and Verhoef (2008) provide empirical evidence that supports this specification. de Palma and Fosgerau (2009) uses an equivalent formulation to the current in which scheduling utility taken to be a general concave function of trip duration and arrival time.

Consider a single individual facing a fixed travel time c, such as in the case when there is no congestion. His scheduling utility as a function of the arrival time at the bottleneck a is then $u(a-c,a)$. Denote the optimal arrival time at the destination by $a_*(c) = \arg\max_a u(a-c,a)$. This exists uniquely by the assumptions on u. Note that the first order condition for utility maximisation is that

$$u_1(a_*(c) - c, a_*(c)) + u_2(a_*(c) - c, a_*(c)) = 0$$

and that differentiating the first order condition with respect to c shows that

$$a'_*(c) = \frac{u_{11} + u_{12}}{u_{11} + 2u_{12} + u_{22}}. \tag{1}$$

Then condition (3) implies that $a'_*(c) > 0$ such that individuals who live further away from the destination will arrive later at the destination when there is no queue. If also condition (4) holds, then $a'_*(c) < 1$ and individuals who live further away from the destination will depart earlier from home.

It is possible that the density of travellers at different distances is so low that there will not be queueing. Therefore a condition is introduced to guarantee that all travellers will be queueing. The condition comes in three versions of different strengths.

Condition 5 $\forall c \in C : a_*(c) < \frac{F(c)}{\psi} + a_*(c_0)$

Condition 6 $\forall c \in C : a'_* (c) < \frac{f(c)}{\psi}$

Condition 7 $\forall c \in C : 1 < \frac{f(c)}{\psi}$

Note here that (7)⇒(6)⇒(7). Each result below refers to one of these conditions.

We will also make use of the following technical lemma.

Lemma 1 *If $a_0 \leq a_* (c_0)$, $a_0 + N/\psi = a_1 \geq a_* (c_1)$, and if there is no queue at time a_1, then, taking the behaviour of all other travellers as given, any traveller will choose arrival time at the bottleneck in the interval $[a_0, a_1]$.*

3 Nash equilibrium

Consider Nash equilibrium in pure strategies, where no individual has incentive to change his time of arrival at the bottleneck.[2] The following theorem provides some basic characteristics of Nash equilibrium.

Theorem 1 *Assume (5). In Nash equilibrium, arrivals at the bottleneck take place during an interval $[a_0, a_1]$, where $a_1 = a_0 + N/\psi$. There is always queue during this interval. Furthermore, $a_0 \leq a_* (c_0)$ and $a_1 \geq a_* (c_1)$.*

So the first traveller arrives at the bottleneck earlier than he would prefer in the absence of congestion and the last traveller arrives later. The no residual queue property holds, i.e., the queue is exactly gone at the time the last traveller arrives at the bottleneck (see de Palma and Fosgerau, 2009). The next theorem states some properties of Nash equilibrium.

Theorem 2 *Assume (1), (2), (3) and (5). Nash equilibrium exists uniquely. In Nash equilibrium, individuals located at distance c from the bottleneck arrive at the bottleneck at time $a(c)$, where $a(c)$ satisfies*

$$a'(c) = -\frac{u_2\left(a(c) - c, \frac{F(c)}{\psi} + a_0\right)}{u_1\left(a(c) - c, \frac{F(c)}{\psi} + a_0\right)} \frac{f(c)}{\psi} > 0. \tag{2}$$

The arrival schedule at the destination is

$$\frac{F(c)}{\psi} + a_0$$

[2] We accept without proof that any Nash equilibrium is characterised by a differentiable function $a(c)$ describing the time of arrival at the bottleneck for a traveller located at distance c.

and the equilibrium scheduling utility is

$$u\left(a(c)-c,\frac{F(c)}{\psi}+a_0\right).$$

The theorem first gives a differential equation for the arrival time $a(c)$ at the bottleneck as a function of distance. The derivative a' is strictly positive which means that the travellers located at greater distances arrive at the bottleneck later. So in Nash equilibrium, travellers sort according to their distance to the bottleneck.

However, it is not the case that travellers located at greater distances also depart earlier. In general, it is not possible to sign the derivative of the departure time $d'(c)=a'(c)-1$. The numerical exercise in section 5 below shows a case where $d'(c)$ is positive at some distances and negative at other.

The sorting property lies behind the expression for the arrival schedule at the destination. Travellers arrive at the bottleneck in sequence sorted according to their distance to the bottleneck and the sequence is preserved by the bottleneck. The first traveller, located at c_0, arrives at the destination at time a_0. The traveller at c arrives at the destination when the $F(c)$ travellers who are located closer have arrived. They take $F(c)/\psi$ time units to pass the bottleneck and so the traveller at c arrives at time $F(c)/\psi+a_0$.

Given the initial condition $a(c_0)=c_0$, (2) describes $a(\cdot)$ uniquely. The proof of the theorem shows that the equilibrium condition $a(c_1)=a(c_0)+N/\psi$ has a unique solution.

The next theorem describes the evolution of the equilibrium queue.

Theorem 3 *Assume (3),(4) and (7). The equilibrium queue length $q(c)=a_0+F(c)/\psi-a(c)$ is quasiconcave as a function of distance c.*

4 Optimal tolling

This section concerns the socially optimal toll. Recall that the toll at time a at the bottleneck is $\tau(a)$.

Theorem 4 *A socially optimal toll exists when (6) holds. Arrivals at the bottleneck take place during an interval $[a_{\tau 0},a_{\tau 1}]$ according to the schedule $a_\tau(c)=\frac{F(c)}{\psi}+a_{\tau 0}$ where $a_{\tau 0}$ is the unique solution to*

$$0=\int_{a_{\tau 0}}^{a_{\tau 0}+N/\psi}\left(u_1(a-c,a)+u_2(a-c,a)\right)da.$$

An optimal toll satisfies

$$\tau'(a_\tau(c)) = u_1(a_\tau(c) - c, a_\tau(c)) + u_2(a_\tau(c) - c, a_\tau(c))$$

where a_τ is the arrival schedule. The optimal toll satisfies $\tau(a_{\tau 0}) = \tau(a_{\tau 1})$ and may be chosen such that $\tau(a_{\tau 0}) = 0$. A socially optimal toll removes exactly the queue. The sequence of arrivals at the destination is unchanged relative to the Nash equilibrium. The arrival schedule at the bottleneck and at the destination is

$$a_\tau(c) = \frac{F(c)}{\psi} + a_{\tau 0},$$

which is a constant shift of the arrival schedule of the Nash equilibrium.

Social welfare is improved by optimal tolling, since queueing is removed. The toll is a transfer from travellers. So it is of interest to examine whether travellers will be better or worse off under optimal tolling, when the use of revenues does not affect travellers.

Theorem 5 *Assume (3) and (6). Consider social optimum implemented by a toll with $\tau(a_{\tau 0}) = \tau(a_{\tau 1}) = 0$. Then the schedule of arrivals at the destination is earlier in social optimum than in Nash equilibrium: $a_{\tau 0} < a_0$. There exists a location c with $a_\tau(c) > a(c)$.*

There is an interval containing c_0 such that all travellers located in this interval are strictly worse off in social optimum than in Nash equilibrium. There is also an interval containing c_1 such that all travellers located in this interval are strictly better off in social optimum than in Nash equilibrium.

5 Numerical illustration

In this section, the theoretical model is illustrated by a numerical simulation. The simulation assumes a continuum of individuals with mass 1. Their scheduling preferences are given by the Vickrey (1973) type of scheduling utility

$$u(d,t) = \int_0^d e^{-s} ds + \int_t^0 e^s ds.$$

The capacity rate of the bottleneck is 0.5 individuals per hour, such that all can pass the bottleneck in two hours. The distribution of distances to the bottleneck is bimodal, composed of two beta distributions, each with mass $1/2$. One has support on $[1, 1.5]$ and the other has support on $[1.5, 2]$. The cumulative distribution of

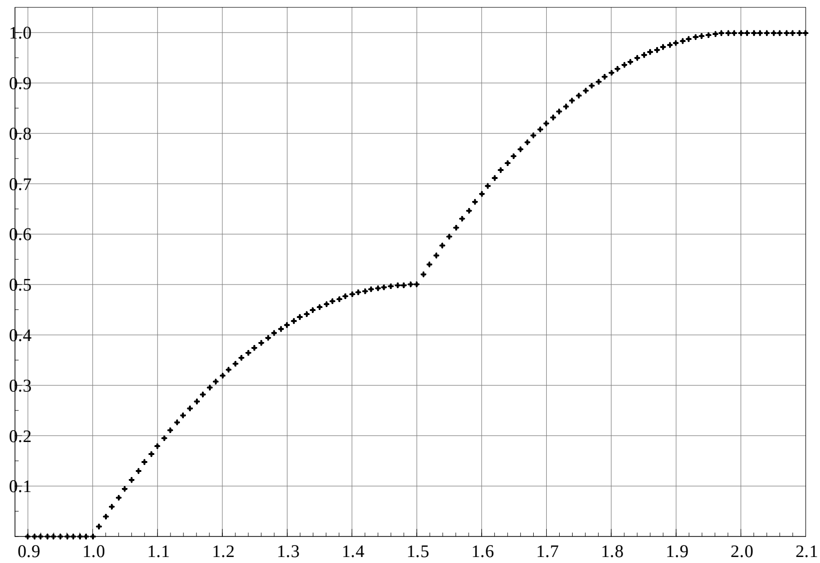


Figure 2: The cumulative distribution of distances to the bottleneck

distances is shown in figure 2. The bimodality of the distribution of distances will be visible in the simulation results.

Given a value of the first arrival time at the bottleneck $a_0 = a(c_0)$, the simulation solves the differential equation (2) numerically to find $a(c)$. Then a search is carried out for the value of a_0 that solves $a(c_1) = a_0 + N/\psi$. The simulation results are shown in figure 3. The figure is discussed in detail below. It shows various functions of the distance to the bottleneck. In addition to the arrival time at the bottleneck, the figure also shows the departure time from home, and the arrival time at the destination. The figure furthermore shows a_*, the preferred arrival time at the destination if there was no queue and it shows the optimum arrival time at the destination.

It is instructive to begin by noting the preferred arrival time a_*, shown in green. The specification of symmetric scheduling utility rates $\beta(t) = \gamma(-t)$ implies that $a_*(c) = c/2$.

Consider now the Nash equilibrium. The first traveller to arrive at the bottleneck is the one located at the least distance c_0. He arrives at the bottleneck at time a_0, which is earlier than $a_*(c_0)$ and arrives at the destination at the same time a_0, since there is no queue for him. Similarly, the last traveller is the one located at the maximum distance c_1. He arrives at the bottleneck at time $a_1 > a_*(c_1)$ and the queue is exactly gone at this time.

Figure 4 shows the duration of the queue as a function of the location of the

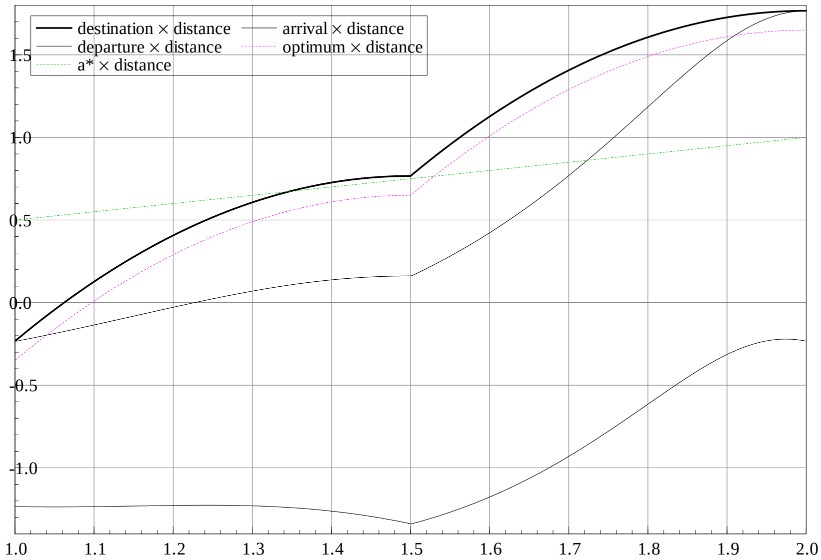


Figure 3: Departure and arrival times

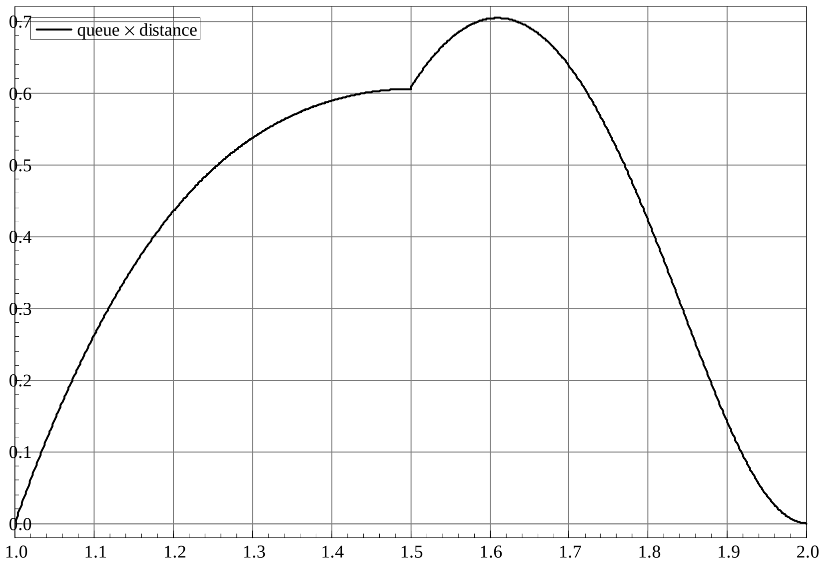


Figure 4: Duration of the queue

travellers. This is not a convex function but it is unimodal as shown in Theorem 3.

Returning to figure 3, consider next the departure time function. This is evidently not a monotone function. In this simulation, the traveller at distance 1.5 departs as the first at time about -1.4. Travellers located closer depart slightly later. For travellers located further away there is almost a monotonous relationship whereby more distant travellers depart later. This confirms the general finding that more distant travellers will not always depart earlier.

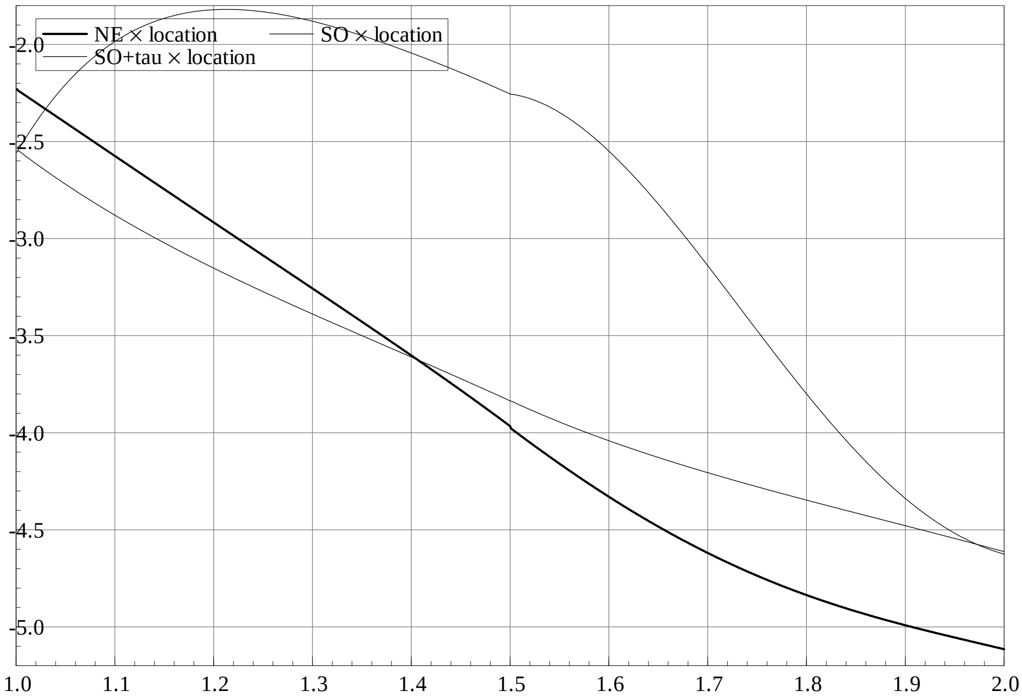
Consider now the social optimum. The arrival time at the destination has the same functional form as the equilibrium arrival time at the destination $F(c)/\psi + a_{\tau 0}$, where $a_{\tau 0}$ is a constant. This happens because the bottleneck capacity is fixed and the sequence of arrivals at the bottleneck is unchanged in optimum relative to equilibrium. In the simulation, $a_{\tau 0}$ is found numerically to maximise average scheduling utility. The pink curve on figure 3 shows the optimum arrival time at the bottleneck as a function of distance. It is also the arrival time at the destination, since there is no queue in optimum. The simulation confirms the result from Theorem 5 that the optimum arrival time is earlier than the equilibrium arrival time. In this case, the first traveller departs about 0.12 hours earlier in optimum than in equilibrium. This means that the traveller at c_0 is worse off in equilibrium since he already arrives before his preferred arrival time. Conversely, the traveller at c_1 is better off.

Figure 5 shows the utilities achieved by individuals at different locations. The fat line shows the indirect utility of individuals in Nash equilibrium, consisting of scheduling utility only. It is decreasing in the distance to the bottleneck. The upper thin line shows the scheduling utility in social optimum. The difference between that and the Nash equilibrium utility, weighted by the density of individuals at different locations, is the welfare gain from tolling. The lower thin line shows the indirect utility in social optimum, equal to the scheduling utility minus the toll. So the toll is visible as the difference between the two thin lines. The indirect utility in social optimum is decreasing but less steeply than in Nash equilibrium.

It is clearly visible how travellers located near the bottleneck lose in social optimum while those far away gain. The indirect utility difference between the individual nearest and furthest from the bottleneck is reduced from 2.9 in Nash equilibrium to 2.1 in social optimum.

6 Conclusion

This paper has introduced spatial heterogeneity into the bottleneck model such that it can be used to represent a city with a central bottleneck. A number of new insights are generated from the model. Perhaps the most important insight is



NE × location
SO × location
SO+tau × location
-2.0
-2.5
-3.0
-3.5
-4.0
-4.5
-5.0
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0

Figure 5: bla

that travellers located near the bottleneck will tend to lose from optimal tolling, while those located far away will tend to gain, when the use of toll revenues is not accounted for.

The crucial property that generates sorting both in equilibrium and optimum is that the schedule of arrivals at the bottleneck a changes with a' having the same sign as $-(u_{11}+u_{12})$.Condition (3) then ensures that $a'>0$. Strict concavity requires that $u_{11}+2u_{12}+u_{22}<0$, but it is still possible to formulate scheduling utility with $u_{11}+u_{12}>0$ for some values of d,t. Future work could investigate the properties of equilibria under such scheduling utility. The illustrative case of $\alpha-\beta-\gamma$ scheduling utility discussed in the Introduction does not fit into this framework as it is piecewise linear and hence not strictly concave.

The spatial distribution of travellers is a source of heterogeneity in the model. It would be of interest to introduce other sources of heterogeneity into the model. One issue would be the robustness of the sorting property. Another kind of extension would be to introduce risk into the model, for example in the form of random capacity (Arnott et al., 1999) or random queue sorting (de Palma and Fosgerau, 2009).

Perhaps the most interesting extension would be to make the location of individuals endogenous. This would tie together the congestion dynamics with urban economic models. For example Arnott (1998) combines a model of urban spatial structure with the $\alpha-\beta-\gamma$ bottleneck model. Optimal tolling does not change transport costs for travellers so when the revenues are not returned, optimal tolling will have no effect on urban structure. As Arnott (1998) notes, this is in contrast to urban economic models with static congestion. However, the Arnott (1998) result is a consequence of space essentially being assumed away in the specification of preferences as was discussed in the Introduction to this paper.

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A Appendix

Lemma 1.
Proof. Consider an arbitrary traveller located at c. Then $a_0 \leq a_* (c_0) \leq a_* (c) \leq a_* (c_1) \leq a_1$. Therefore the traveller prefers to arrive at the bottleneck at time a_0 to any time before, since there is no queue at time a_0. Similarly, he prefers arriving at the bottleneck at time a_1 to any time after, since the queue is gone at time a_1. Therefore he will choose to arrive at the bottleneck during $[a_0, a_1]$. ■

A.1 Nash equilibrium

Lemma 2 *The arrival rate given by (2) and (6) satisfies $a (c_0) \leq a_* (c_0)$ and $a (c_1) \geq a_* (c_1)$ when (5) holds.*

Proof. Note first that $a (c) > a_* (c)$ implies that $a' (c) > f (c) /\psi$. So it is not possible to have $a (c) > a_* (c)$ for all c, since then $a_1 - a_0 > N/\psi$ contradicting (6).

Assume now that $a_0 > a_* (c_0)$. Let c' be the first c with $a (c) = a_* (c)$. Then $a (c') > a_0 + F (c) /\psi > a_* (c_0) + F (c) /\psi > a_* (c')$, which is a contradiction. Hence $a (c_0) \leq a_* (c_0)$ follows.

Assume now that $a_1 < a_* (c_1)$. If $a_0 + F (c) /\psi < a_* (c)$ for all c then $a' (c) < f (c) /\psi$ for all c, which is a contradiction with (6). So there is a last c'' with $a_0 + F (c'') /\psi = a_* (c'')$. Now $N/\psi - F (c'') /\psi = a_1 - a_* (c'') < a_* (c_1) - a_* (c'') < N/\psi - F (c'') /\psi$, by (5). This is a contradiction and hence $a (c_1) \geq a_* (c_1)$ follows. ■

Theorem 1.

Proof. Note first that $a_*(c)$ is strictly increasing in c such that $a_*(c_0) < a_*(c_1)$. Consider Nash equilibrium and let $[a_0, a_1]$ be the smallest interval containing all arrivals at the bottleneck. Then $a_0 \leq a_*(c)$ for all c, since otherwise it would be possible for some to postpone arrival at the queue without meeting congestion and increase utility. Hence $a_0 \leq a_*(c_0)$. The argument for $a_1 \geq a_*(c_1)$ is similar. If $a_1 - a_0 > N/\psi$, then there will exist an interval where the bottleneck capacity is not fully utilised and where it will be possible for some to relocate to increase utility, since $a_*(c_1) - a_*(c_0) < N/\psi$ by assumption. This would contradict Nash equilibrium. If $a_1 - a_0 < N/\psi$, then there is a residual queue at time a_1 and the last individual to arrive could postpone departure from the destination without delaying arrival. This would lead to a strict increase in utility which would contradict Nash equilibrium. Hence $a_1 = a_0 + N/\psi$. ■

Theorem 2.

Proof. Assume first that Nash equilibrium exists. By Theorem 1, there is always queue during $[a_0, a_1]$, such that $R(a) \geq \psi(a - a_0)$. The first order condition for utility maximisation for an individual located at distance c is

$$0 = u_1\left(a - c, \frac{R(a)}{\psi} + a_0\right) + u_2\left(a - c, \frac{R(a)}{\psi} + a_0\right)\frac{\rho(a)}{\psi} \tag{3}$$

and the corresponding second order condition is (suppressing some notation)

$$0 > u_{11} + 2u_{12}\frac{\rho}{\psi} + u_{22}\left(\frac{\rho}{\psi}\right)^2 + u_2\frac{\rho'}{\psi}. \tag{4}$$

Denote the solution by $a(c)$. Achieved utility $u\left(a(c) - c, \frac{R(a(c))}{\psi} + a_0\right)$ for an individual at c satisfies

$$\begin{aligned}
\frac{\partial}{\partial c}u\left(a(c) - c, \frac{R(a(c))}{\psi} + a_0\right) &= u_1 \cdot (a' - 1) + u_2\frac{\rho}{\psi}a' \\
&= -u_1 < 0
\end{aligned}$$

by the first order condition (3). Then utility is decreasing in the distance from the bottleneck.

Differentiate the first order condition (3) with respect to c to find that

$$\begin{aligned}
0 &= u_{11} \cdot (a' - 1) + u_{12} \cdot \frac{\rho}{\psi}(2a' - 1) + u_{22} \cdot \frac{\rho^2}{\psi^2} \cdot a' + u_2 \cdot \frac{\rho}{\psi}a' \\
&= \frac{\partial^2 u}{\partial a^2} \cdot a'(c) - (u_{11} + u_{12}).
\end{aligned}$$

By Condition (3), $a'(c) > 0$. Then $a(\cdot)$ has an inverse $c(\cdot)$ with $a(c(a)) = a$ and $c'(a) = 1/a'(c(a)) > 0$. In this case, $R(a) = F(c(a))$, such that

$$\rho(a) = \frac{f(c(a))}{a'(c(a))}. \tag{5}$$

The first order condition (3) then shows that

$$a'(c) = \frac{f(c)}{\rho(a(c))} = -\frac{u_2\left(a(c) - c, \frac{F(c)}{\psi} + a_0\right)}{u_1\left(a(c) - c, \frac{F(c)}{\psi} + a_0\right)} \frac{f(c)}{\psi},$$

such that $a(\cdot) > 0$ is determined from $a(c_0) = a_0$ by $a(c) = a_0 + \int_{c_0}^{c} a'(\zeta)\, d\zeta$.

Equilibrium requires that $a(c_1) = a_0 + N/\psi$. This defines a_0 uniquely as the following argument shows. Note first that

$$\begin{aligned} \frac{\partial a'(c)}{\partial a_0} &= \left[\frac{u_2}{u_1^2}\left(u_{11}\frac{\partial a(c)}{\partial a_0} + u_{12}\right) - \frac{u_{12}\frac{\partial a(c)}{\partial a_0} + u_{22}}{u_1}\right]\frac{f(c)}{\psi} \\ &= \frac{u_2}{u_1^2}\left[\left(u_{11} - \frac{u_1}{u_2}u_{12}\right)\left(\frac{\partial(a(c) - a_0)}{\partial a_0} + 1\right) + \left(u_{12} - \frac{u_1}{u_2}u_{22}\right)\right]\frac{f(c)}{\psi}. \end{aligned}$$

This is strictly positive by conditions (1) and (2) if $\frac{\partial(a(c)-a_0)}{\partial a_0} \geq 0$. Note next that

$$\frac{\partial(a(c) - a_0)}{\partial a_0} = \int_{c_0}^{c} \frac{\partial a'(c)}{\partial a_0} dc.$$

Then $\frac{\partial(a(c)-a_0)}{\partial a_0} > 0$ if $\frac{\partial(a(\zeta)-a_0)}{\partial a_0} \geq 0$ for all $\zeta < c$. Also, $\frac{\partial a(c_0)-a_0}{\partial a_0} = 0$ and $\frac{\partial a'(c_0)}{\partial a_0} > 0$ such that $\frac{\partial(a(c)-a_0)}{\partial a_0} \geq 0$ for c in a small neighbourhood around c_0. Therefore $\frac{\partial(a(c)-a_0)}{\partial a_0} > 0$ for all $c > c_0$. In particular, $\frac{\partial(a(c_1)-a_0)}{\partial a_0} > 0$. Since equilibrium requires that $a(c_1) - a_0 = N/\psi$, there can only be one equilibrium.

It remains to show that equilibrium exists, i.e. that there exists a_0 such that $a(c_1) - a_0 = N/\psi$. It is sufficient to show that there are values of a_0 such that $\int_{c_0}^{c_1} a'(c)\, dc$ can attain values both larger and smaller than N/ψ. Consider therefore first $a_0 + N/\psi < a_*(c_0)$. Then by (1) and (2),

$$a(c) \leq \frac{F(c)}{\psi} + a_0$$

implies that

$$a'(c) = -\frac{u_2\left(a(c) - c, \frac{F(c)}{\psi} + a_0\right)}{u_1\left(a(c) - c, \frac{F(c)}{\psi} + a_0\right)} < 1.$$

Moreover, (A.1) holds near c_0 and therefore both inequalities hold for all c. But then

$$\int_{c_0}^{c_1} a'(c)\, dc < \int_{c_0}^{c_1} \frac{f(c)}{\psi} dc = \frac{N}{\psi}.$$

The opposite inequality can be obtained for $a_0 > a_*(c_1)$. This establishes existence of equilibrium.

We have shown that (2) together with

$$\int_{c_0}^{c_1} a'(c)\, dc = N/\psi \tag{6}$$

has a unique solution. This does not depend on the existence of equilibrium. Using the above arguments, it is then easy to see that no individual can improve his arrival time at the bottleneck within $[a_0, a_1]$. It remains to be shown that $a(c_0) \leq a_*(c_0)$ and $a(c_1) \geq a_*(c_1)$. But this is shown in Appendix Lemma 2. Therefore, by Lemma 1, the proposed solution does indeed define a Nash equilibrium, which then exists uniquely. ■

Theorem 3.
Proof. The equilibrium queue length has derivative

$$q'(c) = \frac{f(c)}{\psi} \frac{u_1\left(a(c) - c, a_0 + \frac{F(c)}{\psi}\right) + u_2\left(a(c) - c, a_0 + \frac{F(c)}{\psi}\right)}{u_1\left(a(c) - c, a_0 + \frac{F(c)}{\psi}\right)}.$$

Notation for the point where u is evaluated is suppressed in the remainder of the proof. The derivative q' has the same sign as $u_1 + u_2$. We shall show that $u_1 + u_2 < 0$ implies that $u_1 + u_2$ is decreasing. This is sufficient to guarantee that q is quasiconcave. So compute

$$\begin{aligned}
\frac{\partial(u_1 + u_2)}{\partial c} &= (u_{11} + u_{12})(a' - 1) + (u_{12} + u_{22})\frac{f}{\psi} \\
&= -(u_{11} + u_{12})\left(\frac{u_2}{u_1}\frac{f}{\psi} + 1\right) + (u_{12} + u_{22})\frac{f}{\psi} \\
&< -(u_{11} + u_{12})\frac{u_1 + u_2}{u_1} + (u_{12} + u_{22})
\end{aligned}$$

where the second equality follows from (2) and the inequality follows from the assumptions of the lemma. Then $u_1 + u_2 < 0 \Longrightarrow \frac{\partial(u_1+u_2)}{\partial c} < 0$. ■

A.2 Optimal tolling

Theorem 4.

Proof. In optimum, arrivals at the bottleneck take place during an interval of length N/ψ. If there is queue at some point, then arrivals can be delayed such that welfare is improved and the queue is reduced. Hence there is no queue in social optimum and $R_\tau(a) = \psi(a - a_{\tau 0}), \rho_\tau(a) = \psi$. Denote by $a_\tau(\cdot)$ the optimal arrival time at the bottleneck under optimal tolling for an individual located at c. The first order condition for utility maximisation is

$$\tau'(a_\tau(c)) = u_1(a_\tau(c) - c, a_\tau(c)) + u_2(a_\tau(c) - c, a_\tau(c))$$

and the second order condition is

$$\tau''(a_\tau(c)) > u_{11}(a_\tau(c) - c, a_\tau(c)) + 2u_{12}(a_\tau(c) - c, a_\tau(c)) + u_{22}(a_\tau(c) - c, a_\tau(c)).$$

Differentiate the first order condition with respect to c to find that

$$u_{11} \cdot (a'_\tau - 1) + u_{12} \cdot (2a'_\tau - 1) + u_{22}a'_\tau - \tau'' \cdot a'_\tau = 0, \tag{7}$$

which implies that $a'_\tau > 0$ by the second order condition and condition (3). Then $R_\tau(a_\tau(c)) = F(c) = \psi(a_\tau(c) - a_{\tau 0})$, such that $a_\tau(c) = \frac{F(c)}{\psi} + a_{\tau 0}$, and the inverse of a_τ is $c_\tau(a) = F^{-1}(\psi(a - a_{\tau 0}))$. So now from the first order condition,

$$\tau'(a) = u_1\left(a - F^{-1}(\psi(a - a_{\tau 0})), a\right) + u_2\left(a - F^{-1}(\psi(a - a_{\tau 0})), a\right), \tag{8}$$

such that,

$$\begin{aligned}
\tau(a) - \tau(a_{\tau 0}) &= \int_{a_{\tau 0}}^{a} \tau'(s)\, ds \\
&= \int_{a_{\tau 0}}^{a} \left(u_1\left(s - F^{-1}(\psi(s - a_{\tau 0})), s\right) + u_2\left(s - F^{-1}(\psi(s - a_{\tau 0})), s\right)\right) ds.
\end{aligned}$$

Any such toll ensures that individuals located at c will prefer to arrive at the bottleneck at time $a_\tau(c)$ to any other time during $[a_{\tau 0}, a_{\tau 1}]$.

The average scheduling utility under tolling is

$$Eu = \int_{c_0}^{c_1} u\left(\frac{F(c)}{\psi} + a_{\tau 0} - c, \frac{F(c)}{\psi} + a_{\tau 0}\right) f(c)\, dc.$$

Differentiate with respect to a_0 to find that (writing u_1 for $u_1\left(\frac{F(c)}{\psi}+a_0-c, \frac{F(c)}{\psi}+a_0\right)$ etc.)

$$\begin{aligned}\frac{\partial Eu}{\partial a_0} &= \int_{c_0}^{c_1}(u_1+u_2)\, f(c)\, dc \\ \frac{\partial^2 Eu}{\partial (a_0)^2} &= \int_{c_0}^{c_1}(u_{11}+2u_{12}+u_{22})\, f(c)\, dc<0.\end{aligned}$$

So Eu is concave as a function of a_0; $\frac{\partial Eu}{\partial a_0}$ is positive for some a_0 sufficiently smaller than 0 and negative for some a_0 sufficiently larger than 0. This follows from the assumption that u_1+u_2 varies over all of $\mathbb{R}$. Then Eu attains a global maximum. Make the change of variable $a=\frac{F(c)}{\psi}+a_{\tau 0}$ in the the equation $\frac{\partial Eu}{\partial a_0}=0$ to find that the optimal location of the interval of arrival at the bottleneck is determined by

$$0=\int_{a_{\tau 0}}^{a_{\tau 0}+N/\psi}(u_1(a-c,a)+u_2(a-c,a))\, da.$$

Then by (8), $\tau(a_{\tau 0})=\tau(a_{\tau 1})$.

There remains the possibility that individuals may prefer a time outside this interval. By Lemma 1, this will not happen if the toll is such that $\tau(a_{\tau 0})=\tau(a_{\tau 1})=0$ and $a_{\tau 0}\leq a_*(c_0), a_*(c_1)\leq a_{\tau 1}$.

To verify that $a_{\tau 0}\leq a_*(c_0), a_*(c_1)\leq a_{\tau 1}$, note that $a(c)\lesseqqgtr a_*(c)\iff \tau'(a)\gtreqqless 0$. The toll is not constant, so there must be a point c' where $a(c')=a_*(c')$. There can only be one such point by (6). The desired conclusion follows.

■

Theorem 5.

Proof. For a person located at $c\in C$, examine the difference in utility between the cases with and without optimal tolling.

$$\Delta u(c)=u(a_\tau(c)-c, a_\tau(c))-\tau(a_\tau(c))-u\left(a(c)-c, \frac{F(c)}{\psi}+a_0\right).$$

Differentiate this expression with respect to c and insert from first order conditions to find that

$$\frac{\partial \Delta u(c)}{\partial c}=u_1\left(a(c)-c, \frac{F(c)}{\psi}+a_0\right)-u_1(a_\tau(c)-c, a_\tau(c)).$$

Therefore the utility difference can be expressed as

$$\Delta u\left(c\right)=\Delta u\left(c_{0}\right)+\int_{c_{0}}^{c}\left[u_{1}\left(a\left(\zeta\right)-c,\frac{F\left(\zeta\right)}{\psi}+a_{0}\right)-u_{1}\left(a_{\tau}\left(\zeta\right)-\zeta,a_{\tau}\left(\zeta\right)\right)\right]d\zeta>0.$$

Note now that $a_0+F\left(c_0\right)/\psi<a_*\left(c_0\right)$ and $a_0+F\left(c_1\right)/\psi>a_*\left(c_1\right)$ and that the curves $a_0+F\left(c\right)/\psi$ and $a_*\left(c\right)$ intersect in the interior of C. Similarly, $a_\tau\left(c_0\right)<a_*\left(c_0\right)$ and $a_\tau\left(c_1\right)>a_*\left(c_1\right)$ and the curves $a_\tau\left(c\right)=a_{\tau0}+F\left(c\right)/\psi$ and $a_*\left(c\right)$ intersect in the interior of C. Thus, $\Delta u\left(c_0\right)\cdot\Delta u\left(c_1\right)<0$, i.e. one of the endpoint differences must be positive and the other must be negative. Which is positive depends on the sign of $a_{\tau0}-a_0$.

It must be the case that $a_{\tau0}<a_0$: Otherwise $a_\tau\left(c\right)\geq a_0+F\left(c\right)/\psi\geq a\left(c\right)$ for all c, which implies that $\Delta u\left(c_0\right)\geq 0$, $\frac{\partial\Delta u(c)}{\partial c}<0$ for interior c by (3) and $\Delta u\left(c_1\right)\leq 0$, which is a contradiction.

Since there is no queue for commuters located at c_0 and c_1, we have $\Delta u\left(c_0\right)<0<\Delta u\left(c_1\right)$. There exists a c with $a_\tau\left(c\right)>a\left(c\right)$: Otherwise the curves $a\left(\cdot\right)$ and $a_\tau\left(\cdot\right)$ do not intersect, which implies that

$$u_{1}\left(a_{\tau}\left(c\right)-c,a_{\tau}\left(c\right)\right)>u_{1}\left(\frac{F\left(c\right)}{\psi}+a_{0}-c,\frac{F\left(c\right)}{\psi}+a_{0}\right)>u_{1}\left(a\left(c\right)-c,\frac{F\left(c\right)}{\psi}+a_{0}\right)$$

for all c. Then $\frac{\partial\Delta u(c)}{\partial c}<0$, which is a contradiction.

The same argument shows that $\frac{\partial\Delta u(c)}{\partial c}|_{c=c_0}<0$ and $\frac{\partial\Delta u(c)}{\partial c}|_{c=c_1}<0$. ■