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Eliciting Utility for (Non)Expected Utility Preferences Using Invariance Transformations

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Abstract

This paper presents a methodology to determine the preferences of an individual facing risk in the framework of (non)-expected utility theory. When individual preference satisfies a given invariance property, his utility function is solution of a functional equation associated to a specific transformation. Conversely, there exist transformations characterizing any given utility function and its invariance property. More precisely, invariance with respect to two transformations uniquely determines the individual utility function. We provide examples of such transformations for CARA or CRRA utility, but also with any other utility specification and discuss the example of DARA and IRRA specifications.

JEL classification: C90, D81.
Keywords: Utility theory; risk aversion, elicitation of preferences.

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1 Introduction

It has been widely accepted for a long time that preferences towards risk matter. Individual preferences towards risk play a crucial role in finance. For example, it is widely agreed that borrowing, saving, retirement or portfolio optimization crucially depend on the attitude towards risk (see Eeckhoudt et al., 2005). From the point of view of financial institutions, the design of standardized portfolios (portfolio positioning) crucially depends on the specification of the utility function and on the distribution of risk aversion parameters in the population of investors.

However, it is only very recently that financial institutions have started to develop questionnaires to measure the level or risk tolerance of their customers, in order to provide a better investment advice. [1] Most of these questionnaires fail to include lottery-type questions necessary to obtain a quantitative measure of the investor's attitude towards risk. As a result, such ordinal measures do not allow to conclude which portfolio best fits investor's preferences. A quantitative analysis, required to obtain such an advice, should involve: (1) the modelling of individual financial decision; (2) the elicitation of the utility function $U(.)$ and of the probability weighting function, which best capture individual's preferences, and (3) the measurement of the relevant parameters of given utility and weighting functions. Requirement (3) has received much attention in the academic literature (see Keeney and Raiffa, 1976; Diecidue et al. 2009).

Large surveys have been administered by public institutions and scholars to better understand the determinants of risk attitudes. These surveys differ from other ones since they involve more precise and quantitative questions: respondents are typically faced with lotteries (either based on some exogenous monetary amounts, or based on some fraction of their wage). Such surveys have been administrated in several countries and in different specific contexts: saving, retirement, investment, or in more general contexts such as health, smoking, car driving or finance. The interested reader is referred, for example, to Arrondel et al. (2005) for France, Dohmen et al. (2005) for Germany, Guiso and Paiella (2005) for Italy, Donkers et al. (2001), Booij and van de Kuilen (2009) for the Netherlands or Barsky, Juster et al. (1997) for the US. These studies allow investigating the determinants of risk aversion and not so much eliciting utility functions which best represent individual preferences, as for Requirement (2). Indeed, the major drawback of this study is that nothing is done to validate the type of utility function which should be used.

Methods for eliciting the utilities are difficult to construct and implement. As a result, many studies assume that the individual's utility function is characterized by a constant relative (or a absolute) risk aversion. Often, a single value for the degree of relative risk aversion is considered. Unfortunately, this homogeneity assumption may lead to misleading conclusions, and even misleading

[1] In Europe, the use of risk questionnaires is compulsory since November 2007 (European MiFID directive, http://ec.europa.eu/internal_market/securities/isd/mifid_en.htm).

estimated parameters. Alternatively, a prespecified distribution of risk aversion is considered. Specific functional forms such as, for example, CRRA or CARA are justified by the tractability of such specifications. However, their use is not innocuous, as illustrated by de Palma and Prigent (2008), who compare different utility specifications for portfolio optimization. In this paper, we would like to go one step further and investigate the rationale for choosing a specific utility specification. We propose an operational way to construct lotteries allowing to test functional forms, which best represent individual preferences.

We present a general method for eliciting utilities in the context of expected[2] and non-expected utility, especially the RDU (for example the weighted utility as well as the cumulative prospect theory of Tversky and Kahneman, 1992). This method allows separating the elicitation of utility function from probability weighting function, without restricting to expected utility maximization. Our approach is therefore an alternative to the trade-off method proposed by Wakker and Deneffe (1996), and Abdellaoui (2000). The main difference concerns the way the method can be used with real data on observed behavior, focussing on the interpretation/modelling of the error term, as illustrated in de Palma et al. (2008).

The basic idea is that, for a given utility functional form, there exist transformations of the lotteries faced by an individual so that if this individual is indifferent between two lotteries, he will be indifferent between the transformations of these two lotteries. A utility functional form can then be identified by a transformation which keeps indifference relationship invariant. That is to say, we wish to find a set of parametrized transformations $\Psi_t(.)$ of the outcomes of lotteries, such that an individual who is indifferent between two lotteries remains indifferent between the corresponding two transformed lotteries if and only if his preference are represented by a utility function $U(.)$. More precisely, we show that indifference with respect to two families of transformations allows to uniquely determine any utility function. We provide an operational way to construct these transformations.

A similar approach has been used by Miyamoto and Wakker (1996) for the special cases of CARA or CRRA utility functions. Considering the CARA utility function, it is easy to see that if an individual is indifferent between two lotteries $\mathcal{L}_1$ and $\mathcal{L}_2$, then he should also be indifferent between $\Psi_t^a(\mathcal{L}_1)$ and, $\Psi_t^a(\mathcal{L}_2)$, where $\Psi_t^a(.)$ denotes the additive transformation of the outcomes of the lotteries, with shift parameter t. An additive transformation acts as an excise tax. Additive invariance means that if an individual is indifferent between two lotteries, s/he will remain indifferent between the same two lotteries shifted by the same amount t. As shown by Keeney and Raiffa 1976), the utility is a linear/exponential function if and only if it is invariant under addition of a constant. Such results for the additive and multiplicative transformations

[2] Expected utility theory has been criticized by some empirical and theoretical studies (see the arguments in favour and against this theory by Allais, 1953, Kahneman and Tversky, 1979, or Epstein and Schneider, 2003).

are also proved for rank-dependent utility and cumulative prospect theory (see Miyamoto 1988; Wakker and Tversky 1993; Miyamoto and Wakker, 1996; Dyckerhoff, 1994; Fishburn, 1995).

However, indifference with respect to the additive transformation $\Psi_t^a(.)$ does not mean that the preferences are unique, and only coherent with the CARA specification. There exists another solution, which is here straightforward to find: the linear utility function corresponding to risk-neutral preferences. The use of a second transformation, for example $\Psi_t^m(.)$, which for example multiplies all outcomes of lotteries by the same constant t, will allow (in case multiplicative invariance is rejected) to reject the second solution (linear utility) and keep the CARA utility function as the unique representation of preferences. A multiplicative transformation acts as inflation. We use lotteries with continuous distributions for two reasons. First, the continuous version of the RDU allows the determination of the utility function directly on the whole range (here $U(x)$, for all $x \in \mathbb{R}^+$). Second, concerning applications in finance, we can propose lotteries corresponding to actual asset returns, which have continuous cdf (lognormal distributions for example).

The paper is organized as follows. In Section 2, we introduce the general result and show that the utility functions which preserve invariance with respect to a family of transformations obey a functional equation (see Theorem 1). In Section 3, we show that two transformations are enough (and needed) to uniquely elicit individual preferences (see Theorem 2). Before the conclusion (Section 5), in Section 4, using the functional equations derived in Section 2, we provide a characterization of several standard utility functions (CARA, CRRA and also a DARA and IRRA utility) and construct the corresponding invariant transformations. We also show that our approach is valid for the CPT case. The proofs of technical results are relegated to the Appendix.

2 The general result

The main result of this section is that the invariance of preferences with respect to transformation of outcomes induce strong restrictions on individual preferences. More precisely, if whenever the individual is indifferent between two lotteries, she is also indifferent between any two lotteries obtained from the previous ones by a given family of transformations, then his utility function must be solution of specific functional equations. These functional equations characterize individual preferences.

We suppose that the individual maximizes an expectation of his utility U with possible modifications of the initial probability distribution. This framework is quite general: it includes for example the weighted utility introduced in Chew (1989), the rank-dependent utility (see Segal, 1989) or the cumulative prospect theory of Tversky and Kahneman, (1992). We consider probability distributions associated to lotteries having density functions (pdf). Indeed, standard weak convergence results, for example basic Monte Carlo simulations (see Prigent, 2003) prove that any lottery with a pdf is the limit of sequences of discrete lotteries. Then, it is easily shown that, if the invariance property is satisfied for any two discrete lotteries, then it must be satisfied for any lotteries associated to pdf. We illustrate the convergence result for the CPT case. Tversky and Kahneman (1992) have introduced on one hand specific utility functions U^- and U^+ for losses and gains, on the other hand transformation functions w^- and w^+ of the cumulative distributions F.

Proposition 1 *The continuous version of the utility V of the lottery for the CPT case is given by:*

$$V(L) = \int_a^{x^*} U^-(x)\, w^{-\prime}[F(x)]f(x)dx + \int_{x^*}^{\infty} U^+(x)\, w^{+'}[1-F(x)]f(x)dx.$$

Proof. See Appendix. ■

In what follows, we provide general results assuming that the weighting function, associated to the probability modification, corresponds to a transformation of the true pdf into another pdf. We consider outcomes that are positive.[3]

Assumption 1 *The weighting function corresponds to a surjective functional $\varphi : \mathcal{P} \longrightarrow \mathcal{P}$, where $\mathcal{P}$ denotes the set of all pdf. This means that:*

- *For any pdf f on $\mathbb{R}^+$, $\varphi(f)$ is positive.*
- *For any pdf f on $\mathbb{R}^+$, $\int_{\mathbb{R}^+} \varphi(f)(x)\, dx = 1$.*
- *For any pdf $\widetilde{f}$ on $\mathbb{R}^+$, there exists a pdf f on $\mathbb{R}^+$ such that $\varphi(f) = \widetilde{f}$.*[4]

This assumption is obviously satisfied for the standard expected utility maximization, since in that case, we simply have $\varphi(f) = f$. This is a quite general assumption which is valid in particular in the CPT case (see Section 4.4).

Proposition 2 *For any utility function $U(.)$, there exists a non-negative function $p(x)$ satisfying $\int_{\mathbb{R}^+} p(x)\, dx = 1$, and such that the function $U(.)$ belongs to the Hilbert space $\mathbb{L}^2(\mathbb{R}^+, \mu(dx))$, where the measure μ is defined by $\mu(dx) = p(x)\, dx$.*

Proof. See Appendix. ■

Consider now a utility function $U(.) : \mathbb{R}^+ \to \mathbb{R}$, which is assumed to be continuous and non-decreasing. We denote by L_i the lottery defined on the set of outcomes $\mathbb{R}^+$ with the probability density function $f_i : L_i = (\mathbb{R}^+,\ f_i)$.

[3] For example, these outcomes are portfolio returns. Instead of $\mathbb{R}^+$, we could also consider any interval I of $\mathbb{R}$, with no empty interior.

[4] More generally, we can suppose that $\varphi(f) = \lambda\widetilde{f}$, where $\widetilde{f}$ is a pdf and λ is a non negative constant. Since the invariance conditions that we examine correspond to equalities of expectations, we can assume that $\lambda = 1$.

Definition 1 *Let* $\mathcal{U}(L_i)$ *define the expected utility of the lottery* L_i *with respect to the utility function* $U(.)$ *under the probability associated to the density* f_i :

$$\mathcal{U}(L_i) = \int_{\mathbb{R}^+} U(x) f_i(x)\, dx.$$

Let us consider the transformation of outcomes ψ_t, where ψ_t denotes a continuous and non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$. The transformation of the lottery L_i with respect to ψ_t is denoted by L_i^t ($L_i^t = \psi_t(L_i)$).

Definition 2 *Preference* $U(.)$ *is invariant with respect to the transformation* ψ_t *if the following condition holds: for any lotteries* L_1 *and* L_2,

$$\mathcal{U}(L_1) = \mathcal{U}(L_2) \Rightarrow \mathcal{U}\left(L_1^t\right) = \mathcal{U}\left(L_2^t\right). \tag{1}$$

Condition (1) is equivalent to:

$$\begin{aligned} \int_{\mathbb{R}^+} U(x)\left[\frac{\varphi(f_1)(x) - \varphi(f_2)(x)}{p(x)}\right] p(x)\, dx &= 0 \\ &\Longrightarrow \\ \int_{\mathbb{R}^+} U(\psi_t(x))\left[\frac{\varphi(f_1) - \varphi(f_2)(x)}{p(x)}\right] p(x)\, dx &= 0. \end{aligned}$$

Now, we begin by establishing a functional analysis lemma (Proposition 3), which allows the characterization of utility invariance by means of orthogonality. Then, we deduce our main general result about characterization of the utility function through invariance with respect to lottery transformations (see Theorem 1).

Proposition 3 *Consider the space* G *that can be spanned from the set of functions, which are differences between transformations of two different density functions:*

$$G = Vect\left[\left\{g \,|\exists f_1 \text{ and } f_2\ ,\ g = \frac{\varphi(f_1) - \varphi(f_2)}{p}\right\}\right].$$

We have:

$$\mathbb{L}^2\left(\mathbb{R}^+, \mu(dx)\right) = G \oplus Vect[\mathbf{1}], \tag{2}$$

where $Vect[\mathbf{1}]$ *denotes the subspace of constant functions.*

Proof. First, note that if $g \in G$, then we have $\int_{\mathbb{R}^+} g(x)\, p(x)\, dx = 0$.

Let h be any function in $\mathbb{L}^2(\mathbb{R}^+, \mu(dx))$. We have the following identity:

$$h = g + \int_{\mathbb{R}^+} h(x)\, p(x)\, dx, \text{ with } g = h - \int_{\mathbb{R}^+} h(x)\, p(x)\, dx.$$

It remains to prove that $g \in G$: by construction, we have:

$$\int_{\mathbb{R}^+} g(x)\, p(x)\, dx = 0. \tag{3}$$

Let $g^+ = Max(g, 0)$ and $g^- = Max(-g, 0)$. Clearly: $g = g^+ - g^-$. Since p is non-negative by assumption, the previous condition (3) implies that:

$$\int_{\mathbb{R}^+} g^+(x)\, p(x)\, dx = \int_{\mathbb{R}^+} g^-(x)\, p(x)\, dx.$$

Let:

$$\varphi(f_1)(x) = \frac{g^+(x)\, p(x)}{\int_{\mathbb{R}^+} g^+(x)\, p(x)\, dx};\ \varphi(f_2)(x) = \frac{g^-(x)\, p(x)}{\int_{\mathbb{R}^+} g^-(x)\, p(x)\, dx}.$$

Then, we have:

$$\varphi(f_1)(x) \geq 0, \varphi(f_2)(x) \geq 0,$$

and

$$\int_{\mathbb{R}^+} \varphi(f_1)(x)\, dx = \int_{\mathbb{R}^+} \varphi(f_2)(x)\, dx = 1.$$

Since

$$g(x) = \left(\int_{\mathbb{R}^+} g^+(u)\, p(u)\, du \right) \frac{(\varphi(f_1)(x) - \varphi(f_2)(x))}{p(x)},$$

it follows that $g \in G$. Therefore, since $h(x) = g(x) + \int_{\mathbb{R}^+} h(u)\, p(u)\, du$, we deduce that $h \in G + Vect[\mathbf{1}]$.

Using the fact that G and $Vect[\mathbf{1}]$ are orthogonal in $\mathbb{L}^2(\mathbb{R}^+, \mu(dx))$, we prove the result (2). ■

We present now our main result.

Theorem 1 *The utility invariance condition with respect to the transformation ψ_t is equivalent to the existence of parameters α_t and β_t such that:*

$$U_t(.) = \alpha_t U(.) + \beta_t. \tag{4}$$

Proof. From (2), we know that for any function $h \in \mathbb{L}^2(\mathbb{R}^+, \mu(dx))$, there exists two functions f_1 and f_2 and two constants λ and η such that $h = \lambda \frac{\varphi(f_1) - \varphi(f_2)}{p} + \eta$, with $\eta = \int_{\mathbb{R}^+} h(x)\, p(x)\, dx$. Therefore, the invariance condition is equivalent to:

$\forall h \in \mathbb{L}^2(\mathbb{R}^+, \mu(dx))$,

$$\int_{\mathbb{R}^+} U(x)(h(x) - \eta) p(x)\, dx = 0$$
$$\Longrightarrow$$
$$\int_{\mathbb{R}^+} U_t(x)(h(x) - \eta) p(x)\, dx = 0.$$

Finally, this is also equivalent to: $\forall h \in \mathbb{L}^2(\mathbb{R}^+, \mu(dx))$

$$\int_{\mathbb{R}^+} \left[U(x) - \eta \int_{\mathbb{R}^+} U(y) p(y)\, dy \right] h(x) p(x)\, dx = 0 \Rightarrow$$
$$\int_{\mathbb{R}^+} \left[U_t(x) - \eta \int_{\mathbb{R}^+} U_t(y) p(y)\, dy \right] h(x) p(x)\, dx = 0.$$

This means that the orthogonal of the subspace spanned by the function $U(x) - \eta \int_{\mathbb{R}^+} U(y) p(y)\, dy$ is included in the orthogonal of the space spanned by the function $U_t(x) - \eta \int_{\mathbb{R}^+} U_t(y) p(y)\, dy$, $\forall t$. Their biorthogonals satisfy the reverse inclusion. Since these subspaces are finite dimensional, they are closed, for the $\mathbb{L}^2(\mathbb{R}^+, \mu(dx))$ topology. Then they are equal to their biorthogonals.

Therefore, the function $U_t(.) - \eta \int_{\mathbb{R}^+} U_t(y) p(y)\, dy$ belongs to the subspace generated by $U(.) - \eta \int_{\mathbb{R}^+} U(y) p(y)\, dy$. Consequently, there exists α_t and β_t such that:

$$U_t(.) = \alpha_t U(.) + \beta_t.$$

■

This theorem shows that, when preferences are invariant with respect to a specific transformation ψ_t, the function $U_t = U \circ \psi_t$ is a linear transformation of the utility U.

We have considered so far a specific transformation ψ_t. We envisage in the next section invariance with respect to families of transformations $(\psi_t)_t$ and show how preferences can then be elicited.

3 General properties of the family of transformations

We have shown in Theorem (1) that if indifference is preserved by the transformation of the outcomes, then the preference measured by U satisfies:

$$U_t\,(.) = U(\psi_t(.)) = \alpha_t U\,(.) + \beta_t.$$

In some cases discussed in next section (about applications), if we have noticed that the individual preferences are invariant with respect to a given family of transformations, then it is possible to elicit $U(.)$ which satisfies the previous equation.

Theorem (1) however does not provide the construction of the family of transformations $(\psi_t)_t$ for which a given utility $U(.)$ is invariant.

A transformation ψ_t is specified by the utility $U(.)$ and the parametrized functions α_t and β_t that we refer to as seeds. Assume that $U\,(.)$ is strictly increasing from $\mathbb{R}^+$ to $\mathbb{R}$. For $U\,(.)\,,\ \alpha_t$ and β_t given, the transformation is unique and defined by:

$$\psi_t(.) = U^{-1}\left[\alpha_t U\,(.) + \beta_t\right]. \tag{5}$$

Invariance with respect to the transformation $\psi_t(.)$ (corresponding to α_t, β_t, $U\,(.)$) implies that the preferences of the individual can be rationalized using the utility function $U\,(.)$.

However, as we show below, it is the case that such utility function is not unique. Let us examine the problem of uniqueness of the solution. Let us consider for example the following seeds: $\alpha_t = 1$, $\beta_t = t$. In this case, the transformation $\psi_t(.)$ satisfies:

$$\psi_t(.) = U^{-1}\left[U\,(.) + t\right]. \tag{6}$$

It can be immediately shown that $U\,(.)$ is a solution (up to a linear transformation) of the following problem: find $V\,(.)$ such that there exist two parametrized functions $\widehat{\alpha}_t$ and $\widehat{\beta}_t$ satisfying:

$$V(\psi_t(.)) = \widehat{\alpha}(t)V\,(.) + \widehat{\beta}(t). \tag{7}$$

Lemma 1 *If the utility function $V(.)$ is solution of the functional equation (7), where the transformation is defined by Relation (5), then either $V = U$, or $V = (exp[cU] - 1)\,/c$, for some constant c (up to a linear function).*

Proof. From (7), we deduce:

$$V(U^{-1}\left[U\,(x) + t\right]) = \widehat{\alpha}(t)V\,(x) + \widehat{\beta}(t).$$

Let $U(0) = V(0) = 0$. Then: $V(U^{-1}\,[t]) = \widehat{\beta}(t)$. Therefore:

$$V(y) = \widehat{\beta}(U(y)). \tag{8}$$

Therefore, we require to identify the function $\widehat{\beta}(.)$ (up to a linear function).

Equation (8) implies:

$$V(U^{-1}\left[U\left(x\right)+t\right])=\widehat{\beta}(U\left(x\right)+t).$$

Therefore:

$$\widehat{\beta}(U\left(x\right)+t)=\widehat{\alpha}(t)\widehat{\beta}(U\left(x\right))+\widehat{\beta}(t). \tag{9}$$

Letting $z=U\left(x\right)$, we get:

$$\widehat{\beta}(z+t)=\widehat{\alpha}(t)\widehat{\beta}(z)+\widehat{\beta}(t). \tag{10}$$

Expression (10) implies also

$$\widehat{\beta}(z+t)=\widehat{\alpha}(z)\widehat{\beta}(t)+\widehat{\beta}(z). \tag{11}$$

Thus, we have: $(\widehat{\alpha}(z)-1)/\widehat{\beta}(z)=(\widehat{\alpha}(t)-1)/\widehat{\beta}(t)$. Therefore, the function $(\widehat{\alpha}(x)-1)/\widehat{\beta}(x)=a$ constant and $\widehat{\alpha}(x)=a\widehat{\beta}(x)+1$. Inserting this expression in (10) leads to:

$$\widehat{\beta}(z+t)=\widehat{\beta}(z)+\widehat{\beta}(t)+a\widehat{\beta}(z)\widehat{\beta}(t). \tag{12}$$

For $a=0$, this functional equation has trivial (continuous) solutions which are the linear functions. In this case, this shows that $V\left(.\right)$ is a linear transformation of $U\left(.\right)$ as required. We show below that this equation admits a regular function for $a\neq 0$.

From Equation (12), we get:

$$\frac{\widehat{\beta}(z+t)-\widehat{\beta}(z)}{t}=\frac{\widehat{\beta}(t)}{t}\left[1+a\widehat{\beta}(z)\right].$$

Letting $t\rightarrow 0^{+}$, we deduce:

$$\widehat{\beta}^{\prime}(z)=\widehat{\beta}^{\prime}(0)\left[1+a\widehat{\beta}(z)\right].$$

The solution of this differential equation is

$$\widehat{\beta}(z)=\frac{\exp\left[a\widehat{\beta}^{\prime}(0)x\right]-b}{a\widehat{\beta}^{\prime}(0)},$$

where b is a constant. Since $V(0)=0$, we have: $b=1$. Clearly, this function is a solution of Equation (12).

Consequently, if a utility function $V(.)$ is solution of the functional equation (7) where the transformation is defined by Relation (5), then either $V=U$, or $V=\left(exp[cU]-1\right)/c$, for some non-negative constant c (up to a linear function). ■

Let us consider now the following seed: $\alpha_t=t$, $\beta_t=0$. In this case, the transformation $\widetilde{\psi}_t(.)$ satisfies:

$$\widetilde{\psi}_t(.)=U^{-1}\left[tU\left(.\right)\right]. \tag{13}$$

Again, it can be shown that $U(.)$ is a solution (up to a linear transformation) of the following problem: find $V(.)$ such that there exist two parametrized functions $\widetilde{\alpha}_t$ and $\widetilde{\beta}_t$ satisfying:

$$V(\widetilde{\psi}_t(.)) = \widetilde{\alpha}(t)V(.) + \widetilde{\beta}(t). \tag{14}$$

Lemma 2 *If a utility function $V(.)$ is solution of the functional equation (14), where the transformation is defined by Relation (13), then either $V = U$, or $V = U^d$, for some non-negative constant d (up to a linear function).*

Proof. From (14), we deduce:

$$V(U^{-1}[tU(x)]) = \widetilde{\alpha}(t)V(x) + \widetilde{\beta}(t). \tag{15}$$

Let $U(0) = V(0) = 0$. Then $\widetilde{\beta}(t)$ is equal to 0. We have also:

$$\widetilde{\alpha}(t) = \frac{V(U^{-1}[tU(1)])}{V(1)}.$$

Therefore:

$$\frac{V(y)}{V(1)} = \widetilde{\alpha}\left[\frac{U(y)}{U(1)}\right]. \tag{16}$$

Let us assume also that $U(1) = V(1) = 1$. Therefore, we require to identify the function $\widetilde{\alpha}(.)$ (up to a linear function).

Equation (16) implies:

$$V(U^{-1}[tU(x)]) = \widetilde{\alpha}(tU(x)). \tag{17}$$

Thus, from Relations (15) and (17), we get:

$$\widetilde{\alpha}(tU(x)) = \widetilde{\alpha}(t)\widetilde{\alpha}[U(x)]. \tag{18}$$

Letting $z = U(x)$, we get:

$$\widetilde{\alpha}(zt) = \widetilde{\alpha}(z)\widetilde{\alpha}(t). \tag{19}$$

Clearly, a regular function $\widetilde{\alpha}$ which is a solution of Equation (19) is a power function. ■

From the two previous lemmas, we deduce:

Theorem 2 *Suppose that indifference is preserved by the two families of transformations: $\psi_t(.) = U^{-1}[U(.) + t]$ and $\widetilde{\psi}_t(.) = U^{-1}[tU(.)]$. Then, up to a linear function, the individual's utility is unique and given by the function $U(.)$.*

In the next section, we consider standard utility functions. For simple and usual utility functions, the transformation can be found by simple inspection of the utility function.

4 Applications

4.1 The additive characterization

The functional solution for the additive case is given by:

Definition 3 *The additive transformation of the lottery L_i with scale parameter a is:*

$$L_i^{a+} = \left(\mathbb{R}^+, f_i^a\right), \text{ with } f_i^a(x) = f_i(x+a).$$

We provide a functional characterization of additive invariance.

Lemma 3 (Additive invariance) *Consider two functions α and β defined on $\mathbb{R}^+$. If there exists a twice differentiable solution G of the functional equation*

$$G(x+a) = \alpha(a) G(x) + \beta(a), \forall a \in \mathbb{R}, \tag{20}$$

then G is given by:

$$\begin{aligned} G(x) &= b\exp(cx) + d, \text{ or} \\ G(x) &= \delta x + \gamma \end{aligned}$$

with b, c, d, δ and γ real.

Additive invariant utility functions are characterized by:

Proposition 4 (Additive invariance) *If an utility function $U(.)$ satisfies: for any different continuous lotteries L_1 and L_2, the property $\mathcal{U}(L_1) = \mathcal{U}(L_2)$ implies the additive invariance condition:*

$$\mathcal{U}(L_1^a) = \mathcal{U}(L_2^a), \forall a \in \mathbb{R}^+, \text{ with } L_1^a \neq L_2^a.$$

Then we have:

$$\begin{aligned} U(x) &= b\exp(cx) + d, \text{ or} \\ U(x) &= \delta x + \gamma, \end{aligned} \tag{21}$$

where b, c, d, β and γ are real constants.

Proof. Let $U_a(x)$ denote the additive transformation of $U(x)$:

$$U_a(x) = U(x+a), a \in \mathbb{R}^+.$$

1) From previous Theorem, the additive invariance condition, $\mathcal{U}(L_1) = \mathcal{U}(L_2)$ $\Rightarrow \mathcal{U}(L_1^a) = \mathcal{U}(L_2^a)$ with $L_1^a \neq L_2^a$ is equivalent to the condition:

$$\forall a, \exists \alpha_a, \exists \beta_a \ |\forall x, \quad U_a(x) = \alpha_a U(x) + \beta_a\ . \tag{22}$$

2) The proof is now completed since by Lemma (3), we know that the solution of this functional equation is given by (21). ∎

We now consider the multiplicative case. We consider that we multiply all rates by the same function, and study under which condition, the choices are unchanged.

4.2 The multiplicative characterization

The functional solution for the multiplicative case is given by:

Definition 4 *The multiplicative transformation of the lottery L_i with scale parameter a is:*

$$L_i^{a+} = \left(\mathbb{R}^+, f_i^a\right), \text{ with } f_i^a(x) = f_i(xa).$$

We provide a functional characterization of multiplicative invariance.

Lemma 4 (Multiplicative invariance) *Consider two functions $\widehat{\alpha}$ and $\widehat{\beta}$ defined on $\mathbb{R}^+$. If there exists a twice differentiable solution H of the functional equation*

$$H(m\ x) = \widehat{\alpha}(m) H(x) + \widehat{\beta}(m), \forall m \in \mathbb{R}^+,$$

then H is given by:

$$\begin{aligned} H(x) &= bx^c + d, \text{ or} \\ H(x) &= \delta x + \gamma \end{aligned} \tag{23}$$

where b, c, d, δ and γ are constant.

The multiplicative invariant utility functions are characterized by:

Proposition 5 (Multiplicative invariance) *Consider two different lotteries, L_1 and L_2 and a utility function $U(.)$ and assume that $\mathcal{U}(L_1) = \mathcal{U}(L_2)$ implies the multiplicative invariance condition:*

$$\mathcal{U}(L_1^m) = \mathcal{U}(L_2^m), \forall m \in \mathbb{R}^+, \text{ with } L_1^m \neq L_2^m.$$

Then we have

$$\begin{aligned} U(x) &= bx^c + d, \text{ or} \\ U(x) &= \delta x + \gamma, \end{aligned} \tag{24}$$

with b, c, d, δ and γ real.

Proof. The proof is similar to the proof in the additive case. Let $U_m(x)$ denote the multiplicative transformation of $U(x)$:

$$U_m(x) = U(mx), m \in \mathbb{R}^+.$$

1) From previous Theorem, the multiplicative invariance condition,

$$\mathcal{U}(L_1) = \mathcal{U}(L_2) \Rightarrow \mathcal{U}(L_1^m) = \mathcal{U}(L_2^m),$$

is equivalent to the condition :

$$\forall m, \exists \widehat{\alpha}_m, \exists \widehat{\beta}_m \ \Big| \forall x, \ \ U_m(x) = \widehat{\alpha}_m U(x) + \widehat{\beta}_m\ . \tag{25}$$

2) To complete the proof, we use Lemma (4), which provides the expression (23) for the solution to this equation. ■

Similarly, it is straightforward to find a transformation which keeps invariant a HARA utility.

4.3 An example of DARA and IRRA utility

Arrow (1953) advocates that utility should at the same time be DARA (Decreasing Absolute Risk Aversion) and IRRA (Increasing Relative Risk Aversion). The reasoning for DARA is that, for a given risk, wealthy investors are not more risk-averse than poorer ones. IRRA implies that when both wealth and risk increase, then the readiness to bear risk should be reduced. More precisely, for the previous standard portfolio problem with two assets, if the utility function $U(.)$ is twice-differentiable and exhibits DARA and IRRA, then the optimal proportion of initial wealth invested in the risky asset is increasing with wealth; but it increases less than proportionally to the increase in wealth.

DARA means:

$$-\frac{U"(x)}{U'(x)} = f(x);\ f(x) > 0,\ f'(x) < 0.$$

IRRA implies that $xf(x)$ is increasing. A function which satisfies these two requirements is: $f(x) = \sqrt[n]{x}$ with $n \in \mathbb{N}^*$. We can recover the utility function as:

$$U(x) = \exp(-\sqrt[n]{x})P_{n-1}(\sqrt[n]{x}), \tag{26}$$

where $P_{n-1}(.)$ is a polynomial function of order $(n-1)$.

Using the seed: $\alpha_t = 1$, $\beta_t = t$, the transformation $\psi_t(.)$ is given by $U^{-1}[U(.) + t]$ with $U(.)$ defined by (26). Using the seed: $\alpha_t = t$, $\beta_t = 0$, the transformation $\widetilde{\psi}_t(.)$ is given by $U^{-1}[tU(.)]$.

4.4 Kahneman and Tversky specification

Tversky and Kahneman (1992) consider standard weighting transformations, as in Quiggin (1982): There exists a level x^*, two non-decreasing functions w^- and w^+ defined from $[0,1]$ to $[0,1]$, with derivatives $w^{-'}$ and $w^{+'}$such that the individual maximizes:

$$\int_0^{x^*} U^-(x)\, w^{-'}[F(x)]f(x)dx + \int_{x^*}^{\infty} U^+(x)\, w^{+'}[1-F(x)]f(x)dx,$$

where F denotes the cumulative distribution function associated to the pdf f. The utility function defined on losses U^- is convex while the utility function U^+ on gains is concave. In that framework, Wakker and Zank (2002) provides a characterization of CPT and power utility U^- for losses and another power utility U^+ for gains. It corresponds to multiplicative invariances. Our approach allows the elicitation of quite general utility functions. Indeed, the weighting function φ satisfies Assumption (1). We can show that:

Proposition 6 *Assumption (1) is satisfied for the weighting transformation of Quiggin (1982).*

Proof. See Appendix. ■

Given this result, our methodology can be applied by introducing separate transformations on losses and gains. The first families of transformations are applied for outcomes smaller than x^*. It allows the determination of U^-. Similarly, transformations on outcomes higher than x^* yield to the characterization of U^+.

5 Summary

In this paper, we provide a method to elicit utility functions in the framework of non-expected utility (RDU, CPT...). We show that one can decide if an individual preference can be represented by a given utility function by asking a set of questions involving the comparison between two lotteries. We propose a method based on an invariance principle to construct such *transformed* lotteries. These lotteries involve the same probabilities of occurrence but transform the initial outcomes. These transformation functions can be inferred from the given utility function, as illustrated for the standard utility functions (CARA, CRRA) and also for DARA and IRRA utilities. This approach can be easily applied empirically. However, it remains to determine what is the accuracy of the result when the individual responds to a finite set of lottery type questions.

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Appendix

Proof of Proposition 1

For discrete lotteries, Tversky and Kahneman (1992) define the utility as follows. Consider two non-decreasing functions w^- and w^+ on $[0,1]$, and utility type functions U^- and U^+. Let L be the lottery $\{(x_1,p_1),...,(x_l,p_l)\}$ with $x_1 < ... < x_m < x^* < x_{m+1} < ... < x_l$. Define Φ^- and Φ^+ by: $\Phi_1^- = w^-(p_1)$ and $\Phi_l^+ = w^+(p_l)$ and set:

$$\begin{array}{rcl} \Phi_i^- & = & w^-\left(\sum_{j=1}^{i} p_j\right) - w^-\left(\sum_{j=1}^{i-1} p_j\right), \forall i \in \{2,...,m\}, \\ \Phi_i^+ & = & w^+\left(\sum_{j=i}^{l} p_j\right) - w^+\left(\sum_{j=i+1}^{l} p_j\right), \forall i \in \{m+1,...,l\}. \end{array} \tag{27}$$

Then, the utility V on lottery L is given by: $V(L) = V^-(L) + V^+(L)$ with

$$V^-(L) = \sum_{i=1}^{m} U^-(x_i)\Phi_i^- \text{ and } V^+(L) = \sum_{i=m+1}^{l} U^+(x_i)\Phi_i^+. \tag{28}$$

Assume now that the probability distribution F has a pdf f, and the functions w^- and w^+ have derivatives $w^{-'}$and $w^{+'}$. Consider for instance a continuous distribution with a cdf F and a non-negative pdf f on its range (a,b). If (a,b) is a bounded interval, introduce the sequence of discrete lotteries with outcomes $x_{n,k} = (b-a)(k/n)$, for $n \in \mathbb{N}^*, k \in \{1,...,n\}$ and $\mathbb{P}_n[x_{n,k}] = F(x_{n,k}) - F(x_{n,k-1})$. If the range is equal to $[a,+\infty[$, consider the sequence of discrete lotteries with outcomes $x_{n,k,m} = a+m+k/n)$, for $m \in \mathbb{N}$, $n \in \mathbb{N}^*, k \in \{1,...,n\}$ and $\mathbb{P}_n[x_{n,k,m}] = F(x_{n,k,m}) - F(x_{n,k-1,m})$.

Then, the sequence of probability distributions $(\mathbb{P}_n)_n$ weakly converges to the probability distribution with cdf F. Using standard Taylors'expansions of w^-, $w^+$and F, we deduce the convergence of $V_n^-(L)$ defined by:

$$V_n^-(L) = \sum_{x_{n,k} \leq x^*} U^-(x_{n,k}) \left[w^-\left(F(x_{n,k})\right) - w^-\left(F(x_{n,k-1})\right)\right]$$

to $\int_a^{x^*} U^-(x)\, w^{-\prime}[F(x)]f(x)dx$ and also the convergence of $V_n^+(L)$ defined by:

$$V_n^+(L) = \sum_{x_{n,k} > x^*} U^+(x_{n,k}) \left[w^+\left(1 - F(x_{n,k-1})\right) - w^+\left(1 - F(x_{n,k})\right)\right]$$

to $\int_{x^*}^{+\infty} U^+(x)\, w^{+'}[1-F(x)]f(x)dx$. The same results hold for the non-bounded range case.

Therefore, we get the continuous version of the CPT:

$$V(L) = \int_a^{x^*} U^-(x)\, w^{-\prime}[F(x)]f(x)dx + \int_{x^*}^{\infty} U^+(x)\, w^{+'}[1 - F(x)]f(x)dx.$$

Proof of Proposition 2

Without loss of generality, we can assume that the utility function $U(.)$ is in the Hilbert space $\mathbb{L}^2(\mathbb{R}, \mu(dx))$ with $\mu(dx) = p(x)\,dx$, where $p(x)$ is non-negative and

$$\int_{\mathbb{R}^+} p(x)\,dx = 1.$$

Let $U(.)$ be a solution of the problem. Let M be a non-negative real number. Consider the following function q defined by:

$$q(x) = \left(\frac{1}{U^2(x)} I_{\{|U(x)| \geq M\}} + I_{\{|U(x)| < M\}}\right) \exp\left[-x^2\right].$$

Clearly, $q(x)$ is non-negative and $\int_{\mathbb{R}^+} q(x)\,dx$ is a finite number that we denote by A. Define the function p by setting:

$$p(x) = q(x)/A.$$

Then $p(x)$ is non-negative and $\int_{\mathbb{R}^+} p(x)\,dx = 1.$

Additionally,

$$\int_{\mathbb{R}^+} U^2(x)p(x)\,dx =$$

$$(1/A)\left[\int_{\{|U(x)| \geq M\}} \exp\left[-x^2\right] dx + \int_{\{|U(x)| < M\}} U^2(x) \exp\left[-x^2\right] dx\right].$$

Therefore, $\int_{\mathbb{R}^+} U^2(x)p(x)\,dx$ is finite.

Note that our method to elicit the utility function does not require the knowledge of the function p, since this latter one is not involved in Relation (4) of Theorem 1.

Proof of Lemma 3

1) Characterization of the function $\alpha(.)$.

By differentiating Equation (20) with respect to x, we deduce

$$G'(x+a) = \alpha(a)\, G'(x) \text{ and } G'(x+a) = \alpha(x)\, G'(a).$$

Therefore: $\alpha(.)/G'(.) = \lambda$, where λ is a constant.

2) Characterization of the function $G'(x)$.

We have:

$$G'(x+a)/G'(x) = \alpha(a) \text{ and } \alpha(a) = \lambda G'(a).$$

Then:

$$G'(x+a) = \lambda G'(x)\, G'(a).$$

If $G'(0) = 0$, then $G'(x) = 0$, $\forall x$, and $G(x) = \gamma$ (constant). If $G'(0) \neq 0$ then $\lambda = 1/G'(0)$. In this case:

$$G'(x+a) = [G'(x)/G'(0)]\, G'(a). \tag{29}$$

3) Solution of Equation (29) with $G'(0) \neq 0$.

From this equation, we deduce that

$$[G'(x+a) - G'(a)]/x = [G'(x) - G'(0)]/x \times [G'(a)/G'(0)].$$

Taking the limit $x \to 0$, we get

$$G''(a) = G'(a)\, [G''(0)/G'(0)].$$

Thus, either $G''(0) = 0$, then $G''(a) = 0$ which implies $G(x) = \delta x + \gamma$, or $G''(0) \neq 0$, and the solution of this standard differential equation is

$$G(x) = b\exp(cx) + d.$$

Proof of Lemma 4

1) Characterization of the function $\widehat{\alpha}(.)$.

By differentiating Equation (23) with respect to x, we deduce

$$mH'(m\ x) = \widehat{\alpha}(m)\, H'(x) \text{ and } xH'(m\ x) = \widehat{\alpha}(x)\, H'(m)\,.$$

Therefore: $\widehat{\alpha}(x) / (xH'(x)) = \mu$, where μ is a constant.

2) Characterization of the function $H'(x)$.

We have:

$$mH'(m\ x) = \widehat{\alpha}(m)\, H'(x) \text{ and } \widehat{\alpha}(m) = \mu m H'(m).$$

Then

$$mH'(m\ x) = \mu m H'(m) H'(x)\,.$$

If $H'(1) = 0$, then $H'(x) = 0$, $\forall x$, and $H(x) = d$ (constant). If $H'(1) \neq 0$ then $\mu = 1/H'(1)$. In this case:

$$H'(mx) = [H'(m)/H'(1)]\, H'(x)\,. \tag{30}$$

3) Solution of Equation (30) with $H'(1) \neq 0$.

From this equation, we deduce that

$$[H'(m\ x) - H'(m)] / (m[x-1])$$

$$= [H'(x) - H'(1)] / [x-1] \times [H'(m) / (mH'(1))]\,.$$

Taking the limit $x \to 1$, we get $H''(m) = [H'(m)/m] \times [H''(1)/H'(1)]$. Either $H''(1) = 0$, then $H''(m) = 0$ which implies $H(x) = \delta x + \gamma.$, or $H''(1) \neq 1$, and the solution of the differential equation $H''(x) = [H'(x)/x] \times (\text{constant})$ is $H(x) = bx^c + d$.

Proof of Proposition 6

In what follows, we consider standard weighting transformations, as in Quiggin (1982).

Examine for example the probability weighting for the cumulative prospect theory. Denote by F the cumulative distribution function associated to f. There exists a level x^*, two non-decreasing functions w^- and w^+ defined from $[0,1]$ to $[0,1]$, with derivatives $w^{-'}$ and $w^{+'}$ such that the individual maximises:

$$\int_0^{x^*} U(x)\, w^{-\prime}[F(x)]f(x)dx + \int_{x^*}^{\infty} U(x)\, w^{+'}[1-F(x)]f(x)dx.$$

As introduced in Quiggin (1982), both functions w^- and w^+ can be chosen as follows:

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}},$$

with for example $\gamma^- = 0,69$ and $\gamma^+ = 0,61$.

Then, the functional $\varphi(.)$ is defined by:

$$\varphi(f)(x) = f(x)\frac{\left[w^{-'}[F(x)]1_{x\leq x^*} + w^{+'}[1-F(x)]1_{x>x^*}\right]}{[w^-[F(x^*)] + w^+[1-F(x^*)]]}. \tag{31}$$

It satisfies Assumption (1): First, $\varphi(f)$ is actually a pdf. Second, the functional $\varphi(.)$ is surjective. Indeed, for any given pdf $\widetilde{f}$, consider the associated cumulative distribution function $\widetilde{F}$. For a given non-negative parameter λ, consider the function F which satisfies:

$$\begin{aligned} F(x) &= \left(w^-\right)^{-1}[\lambda\widetilde{F}(x)] \text{ for } x \leq x^*, \\ &\text{and} \\ F(x) &= 1 - \left(w^+\right)^{-1}\left(\lambda[1-\widetilde{F}(x)]\right) \text{ for } x > x^*, \end{aligned}$$

where $\left(w^-\right)^{-1}$ and $\left(w^-\right)^{-1}$respectively denote the inverse of the functions w^- and w^+.

We choose $F(x^*)$ such that, for $\lambda = w^-[F(x^*)] + w^+[1-F(x^*)]$, we have:

$$\left(w^-\right)^{-1}[\lambda\widetilde{F}(x^*)] + \left(w^+\right)^{-1}\left(\lambda[1-\widetilde{F}(x^*)]\right) = 1.$$

From the properties of the functions w^- and w^+, it is easy to check that the function F is a differentiable cdf. Denote f its derivative. From Equation (31), we deduce that $\varphi(f) = \widetilde{f}$. Therefore, $\varphi(.)$ is surjective.