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Décembre 2003

Cahier n° 2003-030

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Elementary Games and Games Whose Correlated Equilibrium Polytope Has Full Dimension

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Résumé: Un jeu est élémentaire s'il admet un équilibre corrélé strict à support plein. Un jeu est plein si le polytope de distributions d'équilibres corrélés à dimension pleine. Tout jeu élémentaire est plein. Nous montrons qu'un jeu plein est élémentaire si et seulement si aucune des contraintes d'incitation définissant les équilibres corrélés n'est vide. Plusieurs caractérisations des jeux pleins sont données. Enfin, nous présentons une méthode permettant de construire des jeux pleins mais non élémentaires.

Abstract: A game is elementary if it has strict correlated equilibrium distributions with full support. A game is full if its correlated equilibrium polytope has full dimension. Any elementary game is full. We show that a full game is elementary if and only if all the correlated equilibrium incentive constraints are nonvacuous. Characterizations of full games are provided and examples are given. Finally, we give a method to build full, nonelementary games.

Mots clés : Equilibres corrélés, jeux élémentaires, polytope

Key Words : Correlated equilibria, elementary games, polytope

Classification JEL: C72

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1 Introduction

Elementary games were introduced by Myerson [3]. A game is elementary if it has correlated equilibrium distributions that satisfy all nonnegativity and incentive constraints with strict inequality. As Myerson points out [3, p186]: "For such elementary games, any player can be motivated to choose any pure strategy with no indifference problems" so "correlated equilibrium refinements that generalize Selten's perfectness concept should be unnecessary." Furthermore, Myerson defines a process called dual reduction [3], [8]. This process, which includes elimination of dominated strategies, allows to reduce finite games into games with fewer strategies in a way that selects among correlated equilibria; that is, any correlated equilibrium distribution of the reduced game can be mapped back to a correlated equilibrium distribution of the original game. By iterative dual reduction, any game is reduced to an elementary game, and then the process stops. More precisely, a game is elementary if and only if it cannot be reduced by dual reduction.

A slightly larger, closely related class of games is the class of games whose correlated equilibrium polytope C has full dimension[1], henceforth called *full games*. Nau et al [4] proved that if a game G is full then there is no Nash equilibrium in the relative interior of C [2], which is not generally true.

The aim of this note is to relate and characterize these two classes of games. The remaining of this note is organized as follow: in the next section, the main definitions and notations are introduced. The link between elementary games and full games is made precise in section 3. The last section and appendix B are devoted to characterizations of full games. These can also be used to characterize elementary games. Finally, a method to build full but nonelementary games is explained in appendix A.

2 Notations and Definitions

2.1 Basic notations

The analysis in this note is restricted to finite games in strategic forms. Let $G = \{I, (S_i)_{i\in I}, (u_i)_{i\in I}\}$ denote a finite game in strategic form: I is the nonempty finite set of players, S_i the nonempty finite set of pure strategies of player i and $u_i : \times_{i\in I} S_i \to \mathbb{R}$ the utility function of player i. The set of (pure) strategy profiles is $S = \times_{i\in I} S_i$; the set of strategy profiles for the players other than i is $S_{-i} = \times_{j\in I-i} S_j$. Pure strategies of player i (resp. strategy profiles; strategy profiles of the players other than i) are denoted s_i or t_i (resp. s; s_{-i}). We may write (t_i, s_{-i}) to denote the strategy profile that differs from s only in that its $i-$component is t_i. Finally, N denotes the cardinal of S and $\Delta(S)$ the set of probability distribution over S.

[1] That is, dimension $N - 1$ where N is the number of pure strategy profiles in the game. See section 2.

[2] Except if G is trivial; that is, if the payoff of the players are independent of their own strategy.

2.2 Correlated equilibrium distribution

The set $\Delta(S)$ is an $N-1$ dimensional simplex, henceforth called *the simplex*. A *correlated strategy* of the players in I is an element of the simplex. Thus $\mu = (\mu(s))_{s\in S}$ is a correlated strategy if:

$$\text{(nonnegativity constraints)} \qquad \mu(s) \geq 0 \quad \forall s \in S \tag{1}$$

$$\text{(normalization constraint)} \qquad \sum_{s\in S} \mu(s) = 1 \tag{2}$$

For $(i, s_i, t_i) \in I \times S_i \times S_i$, let h_{s_i,t_i} denote the linear form on $\mathbb{R}^S$ which maps $x = (x(s))_{s\in S}$ to

$$h_{s_i,t_i}(x) = \sum_{s_{-i}\in S_{-i}} x(s)[u_i(s) - u_i(t_i, s_{-i})]$$

A correlated strategy μ is a *correlated equilibrium distribution* [1] if:

$$\text{(incentive constraints)} \qquad h_{s_i,t_i}(\mu) \geq 0 \quad \forall i \in I, \forall s_i \in S_i, \forall t_i \in S_i\backslash\{s_i\} \tag{3}$$

Since conditions (1), (2) and (3) are all linear, the set of correlated equilibrium distributions is a polytope. This polytope, which we denote by C, is a subset of the simplex. Therefore, it has at most dimension $N-1$.

Definition 2.1 *The polytope C has* full dimension *if it has dimension $N-1$.*

2.3 Full games

Definition 2.2 *G is a* full game *if C has full dimension.*

To state more precisely the result of Nau et al [4] mentionned in the introduction, we need some definitions:

Definition 2.3 *Let $(i, s_i, t_i) \in I \times S_i \times S_i$, with $s_i \neq t_i$. The incentive constraint $h_{s_i,t_i}(.) \geq 0$ is* vacuous *if $h_{s_i,t_i} = 0$. That is, if $u_i(s_i, .) = u_i(t_i, .)$.*

Definition 2.4 *A game is nontrivial if at least one of the incentive constraints is non-vacuous: $\exists i \in I, \exists s_i \in S_i, \exists t_i \neq s_i, u_i(s_i, .) \neq u_i(t_i, .)$.*

Nau et al [4] proved that if G is nontrivial, then all Nash equilibria lie on the boundary of C [3]. If furthermore C has full dimension, its boundary coincides with its relative boundary, hence all Nash equilibria lie on its relative boundary. In contrast, if C has less than full dimension, it consists entirely of boundary; the above result is then void and examples of nontrivial games with Nash equilibria in the relative interior of C have actually been found [4].

[3] We could see C as a subset of $\mathbb{R}^N$, in which case C (and $\Delta(S)$) would always have an empty interior. Rather, we see C as a subset of the hyperplane containing the simplex. Therefore, a correlated equilibrium distribution belongs to the *boundary* of C if and only if it belongs to a face of C whose dimension is at most $N-2$.

2.4 Elementary games

Definition 2.5 *A game is elementary [3] if it has correlated equilibrium distributions which satisfy all incentive constraints (3) with strict inequality. That is,*

$$\exists \mu \in C, \forall i \in I, \forall s_i \in S_i, \forall t_i \in S_i - s_i, h_{s_i,t_i}(\mu) > 0 \tag{4}$$

Note that (4) and (1) jointly imply that every pure strategy must have positive marginal probability in μ; that is, $\sum_{s_{-i} \in S_{-i}} \mu(s) > 0 \ \ \forall i \in I, \forall s_i \in S_{-i}$. Also, if some player i is indifferent between two pure strategies s_i and $t_i \neq s_i$ (that is, if $u_i(s_i, .) = u_i(t_i, .)$) then $h_{s_i,t_i}(\mu) = 0$ for all μ in $\Delta(S)$, and (4) cannot be satisfied. Therefore:

Remark 2.6 *If a game is elementary, then all incentive constraints are nonvacuous:* $\forall i \in I, \forall s_i \in S_i, \forall t_i \in S_i \backslash \{s_i\}, u_i(s_i, .) \neq u_i(t_i, .)$.

For comments and results on elementary games, see (Myerson, [3]).

3 The relation between elementary games and full games

In this section, we first give necessary and sufficient conditions for a game to be full. We then precise the link between elementary games and full games.

Proposition 3.1 *The following properties are equivalent:*

(i) *C has full dimension*

(ii) *There exists a correlated equilibrium distribution that satisfies all the nonvacuous incentive constraints with strict inequality. Formally,*

$$\exists \mu \in C, \forall i \in I, \forall s_i \in S_i, \forall t_i \in S_i \backslash \{s_i\}, h_{s_i,t_i} \neq 0 \Rightarrow h_{s_i,t_i}(\mu) > 0$$

(iii) *There exists a correlated equilibrium distribution that satisfies all nonnegativity and nonvacuous incentive constraints with strict inequality.*

Proof. $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are clear. Let us prove $(i) \Rightarrow (ii)$ by contraposition. By convexity of C, (ii) is equivalent to:

$$\forall i \in I, \forall s_i \in S_i, \forall t_i \in S_i \backslash \{s_i\}, h_{s_i,t_i} \neq 0 \Rightarrow (\exists \mu \in C, h_{s_i,t_i}(\mu) > 0)$$

Therefore, if (ii) does not hold, then there exists a nonvacuous incentive constraint that is binding in all correlated equilibrium distributions; this constraint defines an hyperplane whose intersection with the simplex has at most dimension $N - 2$ and includes C; therefore C has at most dimension $N - 2$, contradicting (i). ■

Corollary 3.2 *G is elementary if and only if (a) none of the incentive constraints is vacuous and (b) C has full dimension.*

Proof. Clear from definition 2.5, remark 2.6 and the equivalence of (i) and (ii) in proposition 3.1. ■

Any trivial game (in the sense of definition 2.4) is a full, nonelementary game. A more subtle example of such a game is the following:

Example 3.3
$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 1,1 & 0,0 \\ 1,0 & 0,1 \end{pmatrix} \end{array}$$

There are four incentive constraints. Two of them are vacuous, hence this game is not elementary. However, the correlated strategy assigning probability $1/2$ to both TL and BR checks the two nonvacuous incentive constraints with strict inequality, so, by proposition 3.1, this game is full.

A general method to build full, nonelementary games is given in appendix A.

4 Characterization of these classes of games

In this section , we provide criteria to determine whether C has full dimension. By corollary 3.2, these criteria can also be used to know if a game is elementary. We end this note with two examples: an elementary game and a nonelementary game.

4.1 Characterizations

The following proposition is based on [2], [6, p.186] and [3]. Let G be nontrivial. Consider the following two-player, zero-sum, auxiliary game Γ: the maximizer chooses a strategy profile s in S; the minimizer chooses a player i in N and a couple of strategy (s'_i, t_i) in $S_i \times S_i$, such that $u_i(s'_i, .) \neq u_i(t_i, .)$.[4] The payoff for the maximizer is $u_i(s) - u_i(t_i, s_{-i})$ if $s'_i = s_i$ and 0 otherwise.

Proposition 4.1 *C has full dimension if and only if the value of the mixed extension of Γ is positive*

Proof. A mixed strategy of the maximizer is a correlated strategy μ of G; the payoff if the minimizer chooses (s'_i, t_i) is $h_{s'_i,t_i}(\mu)$. Thus, μ guarantees a positive payoff if and only if μ checks all nonvacuous incentive constraints with strict inequality (and if it does $\mu \in C$). Then apply proposition 3.1. ■

The following propositions apply only to games with a correlated equilibrium distribution with full support (for instance, a completely mixed Nash equilibrium). Let m be a positive integer and $h_1, ..., h_m$ denote the linear forms associated with the nonvacuous incentive constraints.

Proposition 4.2 *Assume that G admits a correlated equilibrium distribution with full support. If $h_1, ..., h_m$ are independent, then C has full dimension.*

Proof. Given in appendix B. ■

If $h_1, ..., h_m$ are not independent, let B be a basis of the linear span of $\{h_1, ..., h_m\}$. Without loss of generality, assume that $B = (h_1, ..., h_q)$ with $1 \leq q < m$. Let

[4] Such a triplet (i, s'_i, t_i) with $u_i(s'_i, .) \neq u_i(t_i, .)$ must exist, because G is nontrivial.

$A = (a_{kl})_{1 \leq k \leq q,\, q+1 \leq l \leq m}$ be the matrix of $(h_{q+1}, ..., h_m)$ in the basis B; that is, for all $q + 1 \leq l \leq n$,

$$h_l = \sum_{1 \leq k \leq q} a_{kl} h_k$$

Let Γ' denote the two-player, zero-sum, auxiliary game, whose payoff matrix for the maximizer is A; that is the maximizer chooses k in $\{1, ..., q\}$, the minimizer chooses l in $\{q + 1, ..., m\}$ and the payoff for the maximizer is a_{kl}.

Proposition 4.3 *Assume that G admits a correlated equilibrium with full support. If $h_1, ..., h_m$ are not independent, C has full dimension if and only if the value of the mixed extension of Γ' is positive.*

Proof. Given in appendix B. ■

The following remarks will be used in the examples:

Remark 4.4 *If in the payoff matrix A of Γ' there is a nonpositive column (resp. all the entries are nonnegative), then the value of the mixed extension of Γ' is nonpositive (resp. positive).*

Proof. The first part is straightforward. For the second part, recall that $h_1, ..., h_m$ are the linear forms associated with the *nonvacuous* incentives constraints. Therefore $h_1, ..., h_m$ are all nonzero. So, for all $q + 1 \leq l \leq m$, there exists $1 \leq k_l \leq q$ such that $a_{k_l l}$ is nonzero. Therefore if all the entries of A are nonnegative then playing a completely mixed strategy guarantees a positive payoff to the maximizer. Hence the value of the mixed extension of Γ' is positive. ■

4.2 Examples

Example 4.5 *An elementary game with linearly dependent incentive constraints.*

The following 3-player, $2 \times 2 \times 2$ game is taken from [4]:

$$\begin{array}{lll} \text{Up:} & & \begin{array}{cc} \text{Left} & \text{Right} \end{array} \\ & \begin{array}{l} \text{Top} \\ \text{Bottom} \end{array} & \begin{pmatrix} 0,0,2 & 0,3,0 \\ 3,0,0 & 0,0,0 \end{pmatrix} \\ \text{Down:} & & \\ & \begin{array}{l} \text{Top} \\ \text{Bottom} \end{array} & \begin{pmatrix} 1,1,0 & 0,0,0 \\ 0,0,0 & 0,0,3 \end{pmatrix} \end{array}$$

This game has a completely mixed Nash equilibrium. There are only five distinct incentive constraints (the constraint for Row defecting from Top to Bottom is the same as the constraint for Column defecting from Left to Right). These five incentive constraints are linearly independent. So the payoff matrix A of the auxiliary game Γ' is a 5×1 column matrix whose entries are four 0 and a 1. So, by proposition 4.3 and remark 4.4, C has dimension 7. Furthermore, none of the incentive constraints is vacuous. Therefore, by corollary 3.2, this game is elementary.

Example 4.6 *A nonelementary game:*

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \left(\begin{array}{cc} 2,-1 & 0,0 \\ 0,0 & 1,-2 \end{array} \right) \end{array}$$

This game has a completely mixed Nash equilibrium. Any incentive constraint is a nonpositive linear combination of the three other incentive constraints, which are linearly independent. So the payoff matrix A of Γ' is a 3×1, nonpositive column matrix. Therefore, by remark 4.4, C has less than dimension 3.[5] In particular, this game is not elementary.

A A method to build full, nonelementary games

We first need a definition:

Definition A.1 *Let* $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ *and* $G' = (I', (S'_i)_{i \in I'}, (u'_i)_{i \in I'})$ *be two finite games.* G' *is* built on G by adding a semi-duplicate to player i *if:*

- $I' = I$
- $S'_j = S_j \ \ \forall j \neq i$
- $\exists t'_i \in S'_i,\ S'_i = S_i \cup \{t'_i\}$
- $u'_k(s) = u_k(s)\ \forall s \in S, \forall k \in I$
- $\exists t_i \in S_i, \forall s_{-i} \in S_{-i},\ u'_i(t'_i, s_{-i}) = u_i(t_i, s_{-i})$ [6]

Example A.2

$$G_1 = \left(\begin{array}{cc} 1,1 & 0,0 \end{array} \right) \quad G'_1 = \left(\begin{array}{cc} 1,1 & 0,0 \\ 1,0 & 0,1 \end{array} \right) \quad G''_1 = \left(\begin{array}{ccc} 1,1 & 0,0 & 0,1 \\ 1,0 & 0,1 & 1,0 \end{array} \right)$$

G'_1 *is built on* G_1 *by adding a semi-duplicate to the row player and* G''_1 *is built on* G'_1 *by adding a semi-duplicate to the column player.*

We can now provide the method:

Proposition A.3 *Let* G *be elementary and* G' *be built on* G *by adding a semi-duplicate to some player. Then* G' *is full and nonelementary.*

[5] More generally, it is easy to prove that if G is nontrivial and best-response equivalent to a two-player zero-sum game [7] (as in example 4.6), C does not have full dimension.

[6] In words, in G' the set of players is the same than in G and the pure strategy sets are the same for all players but i, who has an additional pure strategy t'_i; when player i does not use his additional strategy the payoffs in G' are the same than in G; furthermore player i is indifferent between his additional strategy and a strategy t_i that was already available in G.

Proof. G' is clearly nonelementary, so we only have to prove that G' is full. Let μ in $\Delta(S)$ check all the incentive contraints of G with strict inequality (in the sense of (4)). Define μ' and ν' in $\Delta(S')$ by:

$$\mu'(s) = \mu(s) \ \ \forall s \in S \ \ ; \ \ \mu'(t_i', s_{-i}) = 0 \ \ \forall s_{-i} \in S_{-i}$$

$$\nu'(s) = 0 \ \ \forall s \in S \ \ ; \ \ \nu'(t_i', s_{-i}) = \frac{1}{\mu(t_i \times S_{-i})}\mu(t_i, s_{-i}) \ \ \forall s_{-i} \in S_{-i}$$

Note that for $\epsilon > 0$ small enough, $\mu'_\epsilon = (1-\epsilon)\mu' + \epsilon\nu'$ is a correlated equilibrium distribution of G' that satisfies all its nonvacuous incentive constraints with strict inequality. Then use proposition 3.1. ■

Note that full, nonelementary games cannot all be built by adding semi-duplicates to an elementary game: G_1' cannot be built in this way.

Note also that if G is full but not elementary, then adding a semi-duplicate to G need not yield a full game. For instance, G_1'' is not full. The point is that adding a new strategy to some player may lift the indifference of some other player between two of her strategies. This shall be clear from proposition A.5, which generalizes proposition A.3. We first need a definition:

Definition A.4 *Let G' be a game built on G by adding a semi-duplicate to player i. G'* preserves indifference *in G if for all $j \neq i$ and all s_j, t_j in S_j:*

$$u_j(s_j, .) = u_j(t_j, .) \Rightarrow u_j'(s_j, .) = u_j'(t_j, .)$$

That is, if player j was indifferent between s_j and t_j in G, she is still indifferent between s_j and t_j in G'.

Proposition A.5 *Let G be full and G' be built on G by adding a semi-duplicate t_i' to player i. If G' preserves indifference in G, G' is full. If G is a two-player game, the converse holds, so that G' is full if and only if G' preserves indifference in G.*

Proof. In G', there are three kinds of incentive constraints: constraints of type: (i) $h'_{s_j,t_j}(.) \geq 0$ with $j \neq i$ or, if $j = i$, $s_i \neq t_i'$ and $t_i \neq t_i'$; (ii) $h'_{s_i,t_i'}(.) \geq 0$ with $s_i \in S_i$; (iii) $h'_{t_i',s_i}(.) \geq 0$ with $s_i \in S_i$. (The prime in h' indicates that we consider incentive constraints of G'.) Since G is full, there exists a correlated strategy μ that checks all the nonvacuous incentive constraints of G with strict inequality. Define μ', ν' and μ'_ϵ as in the proof of proposition A.3. We now show that for ϵ small enough, μ'_ϵ satisfies with strict inequality all the nonvacuous incentive constraints of G', which implies that G' is full.

First, for ϵ small enough, μ'_ϵ satisfies with strict inequality all the incentive constraints of type (i) corresponding to incentive constraints of G satisfied by μ with strict inequality. Since G' preserves indifference in G, the other incentive constraints of type (i) are vacuous. Since for all $s_i \in S_i$, $h'_{s_i,t_i'} = h'_{s_i,t_i}$, the above argument also takes care of constraints of type (ii). Finally, the conditional probabilities on S_{-i} given t_i' in μ'_ϵ are the same than the conditional probabilities given t_i in μ. Since $u_i'(t_i', .) = u_i(t_i, .)$, this makes sure that μ'_ϵ satisfies with strict inequality all the nonvacuous incentive constraints of type (iii).

Now assume that G is a 2-player game and that $i = 2$. Let t'_2 be the strategy added to player 2 in G'. If G' does not preserve indifference in G, then there exists $s_1, t_1 \in S_1$ such that player 1 is indifferent between s_1 and t_1 in G but not in G': $u_1(s_1, s_2) \neq u_1(t_1, s_2)$ for all s_2 in S_2 but $u_1(s_1, t'_2) \neq u_1(t_1, t'_2)$. Assume for instance $u_1(s_1, t'_2) > u_1(t_1, t'_2)$; then, in G', s_1 weakly dominates t_1. So the incentive constraint $h'_{t_1,s_1}(.) \geq 0$, which is nonvacuous, cannot be satisfied with strict inequality. Therefore G' cannot be full. ■

B Proof of propositions 4.2 and 4.3

We begin with a claim:

Claim B.1 *C has full dimension if and only if (α) there exists a correlated equilibrium distribution μ with full support and (β) there exists x in $\mathbb{R}^S$ such that x satisfies all nonvacuous incentive constraints with strict inequality.*

Proof. Necessity: follows from proposition 3.1; sufficiency: assume that (α) and (β) hold; let $\nu = (1-\epsilon)\mu + \epsilon x$. For ϵ positive small enough, normalizing ν yields a correlated equilibrium distribution which satisfies all nonvacuous incentive constraints with strict inequality. Then apply proposition 3.1 ■

Claim B.1 implies that if there exists some correlated equilibrium with full support, C has full dimension if and only if (β) holds. We now show that the condition required on top of (α) in proposition 4.2 (resp. proposition 4.3) imply (resp. is equivalent to) condition (β). We will use the following standard result:

Lemma B.2 *Let E be a finite dimensional real vector space, q a positive integer, and $f_1, ..., f_q$ linear forms on E. Then $f_1, ..., f_q$ are linearly independent if and only if for any y in $\mathbb{R}^q$ there exists x in E such that $y = (f_1(x), ..., f_q(x))$.*

The notations below are taken from section 4.1. Assume that $h_1, ..., h_m$ are linearly independent; lemma B.2 then implies that (β) holds, proving proposition 4.2. Assume now that $B = (h_1, ..., h_q)$ is a basis of the linear span of $\{h_1, ..., h_m\}$, for some $1 \leq q < m$. The value of the auxiliary game of proposition 4.3 is positive if and only if

$$\exists y \in \mathbb{R}^q, y \geq 0, \sum_{k=1}^{q} y_k = 1, yA > 0 \tag{5}$$

For x in $\mathbb{R}^S$, let $y(x) = (h_1(x), ..., h_q(x))$. By definition of the matrix A:

$$(h_{q+1}(x), ..., h_m(x)) = y(x)A$$

Therefore (β) holds if and only if there exists x in $\mathbb{R}^S$ such that $y(x) > 0$ and $y(x)A > 0$. But, by lemma B.2, $y(x)$ may be given any value in $\mathbb{R}^q$ by an appropriate choice of x. Therefore (β) is equivalent to:

$$\exists y \in \mathbb{R}^q, y > 0, yA > 0 \tag{6}$$

It is easy to see that (6) is equivalent to (5), completing the proof of proposition 4.3.

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