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Yannick Viossat

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LABORATOIRE D'ECONOMETRIE

1rue Descartes F-75005 Paris

(33) 1 55558215

http://ceco.polytechnique.fr/

mailto:labecox@poly.polytechnique.fr

Replicator Dynamics and Correlated Equilibrium

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Résumé: Cette note établit que dans tous les jeux 3-3 symmetriques, la dynamique des réplicateurs élimine toutes les stratégies qui ne sont utilisées dans aucun équilibre corrélé. Ce résultat s'étend à la dynamique de meilleure réponse et à toutes les dynamiques convexes monotones. La preuve repose sur des arguments de réduction duale.

Abstract: This note establishes that in every 3-3 symmetric game, the replicator dynamics eliminates all strategies that are never used in correlated equilibrium. This extends to the best-response dynamics and to any convex monotonic dynamics. The proof is based on dual reduction.

Mots clés : Equilibre corrélé, dynamiques de jeux, réduction duale

Key Words : Correlated equilibrium, game-dynamics, dual reduction

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[1] Laboratoire d'Econométrie, CNRS et Ecole Polytechnique

1 Notations, definitions and main result

1.1 Notations

This note focuses on finite, two-player symmetric games. Such a game G is given by a set $S = \{1, ..., N\}$ of pure strategies (the same for each player) and a payoff matrix $\mathbf{U} = (U(i,j))_{1\leq i,j\leq N}$. Here $U(i,j)$ is the payoff of a player playing strategy i against a player playing strategy j. Since the game is symmetric, whether the player playing i is called player 1 or player 2 is unimportant.

Let $\Delta(S)$ denote the set of probabilities over S or mixed strategies:

$$\Delta(S) := \left\{ \mathbf{x} \in \mathbb{R}^S : x_i \geq 0 \ \forall i \in S, \sum_{i\in S} x_i = 1 \right\}$$

The payoff of a player playing the mixed strategy **x** against a player playing the mixed strategy **y** will be denoted

$$U(\mathbf{x}, \mathbf{y}) := \sum_{i\in S, j\in S} x_i y_j U(i,j)$$

1.2 Replicator Dynamics

Given some initial condition $\mathbf{x}(0)$ in $\Delta(S)$, the *single population replicator dynamics* is given by

$$\dot{x}_i = x_i \left[U(i,\mathbf{x}) - U(\mathbf{x},\mathbf{x})\right]$$

where $\dot{x}_i$, x_i and $\mathbf{x}$ are taken at time t.

We now define the *two-population* replicator dynamics. Since there are two populations, the set of pure strategies of player 1 is a priori different from the set S_2 of pure strategies of player 2; but since we only consider symmetric games, we may assimilate S_1 and S_2 and write S for both S_1 and S_2. The same remark holds for payoff matrices and we write U for both U_1 and U_2.

Given an initial condition $(\mathbf{x}(0), \mathbf{y}(0))$ in $\Delta(S) \times \Delta(S)$, the *two-population replicator dynamics* is given by:

$$\dot{x}_i = x_i \left[U(i,\mathbf{y}) - U(\mathbf{x},\mathbf{y})\right] \text{ and } \dot{y}_i = y_i \left[U(i,\mathbf{x}) - U(\mathbf{y},\mathbf{x})\right] \tag{1}$$

(here **x** (resp. **y**) represents the mean strategy in the population of players 1 (resp. 2))

Note that, for symmetric games and from a mathematical point of view, the single population replicator dynamics corresponds to the two-population replicator dynamics with symmetric initial conditions (that is, with $\mathbf{x}(0) = \mathbf{y}(0)$).

Definition *A pure strategy i in S of player 1 (resp. player 2) is* eliminated *by the two-population replicator dynamics (for some initial condition $(\mathbf{x}(0), \mathbf{y}(0))$) if $x_i(t)$ (resp. $y_i(t)$) goes to zero as t goes to infinity.*

Note that, if the initial condition is symmetric (or, equivalently, in the single population framework), then the pure strategy i of player 1 is eliminated if and only if the pure strategy i of player 2 is eliminated.

1.3 Correlated equilibrium

A correlated strategy is a probability distribution on the set $S \times S$ of pure strategy profiles. Hence $\mu = (\mu(s))_{s \in S}$ is a correlated strategy if

$$\mu(i,j) \geq 0, \quad \forall (i,j) \in S \times S$$

and

$$\sum_{(i,j) \in S \times S} \mu(i,j) = 1)$$

A correlated strategy is a *correlated equilibrium distribution* (Aumann, 1974) if it satisfies the following incentive constraints:

$$\sum_{j \in S} \mu(i,j)[U(i,j) - U(i',j)] \geq 0 \quad \forall i \in S, \forall i' \in S$$

and symmetrically

$$\sum_{i \in S} \mu(i,j)[U(j,i) - U(j',i)] \geq 0 \quad \forall j \in S, \forall j' \in S$$

The Nash equilibria exactly correspond to the correlated equilibrium distributions μ that are independent; that is, such that there exists mixed strategies **x** and **y** in $\Delta(S)$ such that: $\mu(i,j) = x_i y_j \ \ \forall (i,j) \in S \times S$.

Definition *The pure strategy i in S (resp. the pure strategy profile (i,j) in $S \times S$) is* used in correlated equilibrium *if there exists a correlated equilibrium distribution μ such that $\sum_{j \in S} \mu(i,j) > 0$ (resp. $\mu(i,j) > 0$).*

Remark 1 *Due to the symmetry of the game, the existence of a correlated equilibrium distribution μ such that $\sum_{j \in S} \mu(i,j) > 0$ is equivalent to the existence of a correlated equilibrium distribution μ' such that $\sum_{j \in S} \mu(j,i) > 0$.*

Thus, when we say that some pure strategy i is used (or not used) in correlated equilibrium, it is unnecessary to specify whether we see this strategy as a strategy of player 1 or as a strategy of player 2. Furthermore, due to the symmetry of the game and to the convexity of the set of correlated equilibrium distributions, a pure strategy is used in correlated equilibrium if and only if it is used in some symmetric equilibrium (i.e. in a correlated equilibrium μ such that $\mu(k,l) = \mu(l,k)$ for every (k,l) in $S \times S$). Thus, we do not have to specify whether we are only interested in symmetric equilibria or not.

1.4 Main result

Definition *An initial condition $(\mathbf{x}(0), \mathbf{y}(0))$ of the two-population replicator dynamics is* interior *if for every pure strategy i in S, both $x_i(0)$ and $y_i(0)$ are positive.*

Proposition *Consider a 3×3 symmetric game. If the pure strategy i is not used in correlated equilibrium, then $x_i(t)$ and $y_i(t)$ both converge to 0 under the two-population replicator dynamics (1), for any interior initial condition $(\boldsymbol{x}(0), \boldsymbol{y}(0))$.*

As a particular case, this implies that:

Corollary *In every 3×3 symmetric game, every pure strategy that is not used in correlated equilibrium is eliminated by the* single *population replicator dynamics, for any interior initial condition (that is, for any initial condition $\mathbf{x}(0)$ in $\Delta(S)$ such that $x_i(0) > 0$ for every i in S).*

2 Proof

2.1 Elements of dual reduction

Let us first recall some properties of dual reduction (Myerson, 1997) on which the proof is based. Dual reduction is defined for any finite game, but, so that no new notations be needed, only two-player symmetric games are considered below. For $k = 1, 2$, let $\alpha_k : S \to \Delta(S)$ denote a transition probability over the set of pure strategies S. The image of a pure strategy i by this mapping is a mixed strategy. Denote by $\alpha_k * i$ this mixed strategy.

The vector $\alpha = (\alpha_1, \alpha_2)$ is a *dual vector* (Myerson, 1997) if for all (i, j) in $S \times S$:

$$[U(\alpha_1 * i, j) - U(i, j)] + [U(\alpha_2 * j, i) - U(j, i)] \geq 0 \tag{2}$$

Such a dual vector is *strong* (Viossat, 2004a) if the inequality (2) is strict for any strategy profile (i, j) that is not used in correlated equilibrium. It follows from (Nau & McCardle, 1990, discussion on page 432 and proposition 2) that there exists a strong dual vector and from a variant of the proof of proposition 5.26 in (Viossat, 2004a) that this strong dual vector may be assumed to be symmetric (i.e. we may assume $\alpha_1 = \alpha_2$). In the remaining of the paper, α denotes such a strong and symmetric dual vector.

Note that if the pure strategy i is not used in correlated equilibrium, then for all j in S, the strategy profile (i, j) is not used in correlated equilibrium. Therefore (recall that α is strong):

$$[U(\alpha * i, j) - U(i, j)] + [U(\alpha * j, i) - U(j, i)] > 0 \tag{3}$$

(Here and in the remaining of the paper, $\alpha * i$ denotes either $\alpha_1 * i$ or $\alpha_2 * i$. Since α is symmetric, i.e. $\alpha_1 = \alpha_2$ this is unambiguous.)

2.2 Properties of the replicator dynamics

The only properties of the replicator dynamics that will be used in the proof are the one given below. Let i, i' and $\mathbf{p}$ denote respectively two pure strategies and a mixed strategy of player 1. Fix an interior initial condition $(\mathbf{x}(0), \mathbf{y}(0))$.

Property 1 *If there exist $\epsilon > 0$ and a time T in $\mathbb{R}$ such that, for all $t \geq T$, $U(i, \boldsymbol{y}(t)) < U(i', \boldsymbol{y}(t)) - \epsilon$, then $\boldsymbol{x}_i(t) \underset{t \to +\infty}{\longrightarrow} 0$.*

Property 2 *If $\boldsymbol{p}$ weakly dominates i and if there exists a pure strategy j in $S_2^+ = \{j \in S, U(\boldsymbol{p}, j) > U(i, j)\}$ such that $y_j(t)$ does not go to zero as time goes to infinity, then $\boldsymbol{x}_i(t) \underset{t \to +\infty}{\longrightarrow} 0$*

Of course, the symmetric properties (i.e. on elimination of strategies of player 2) hold just as well. The fact that the replicator dynamics checks properties 1 and 2 is proved in the appendix. Property 2 imply the following better known property:

Property 3 *If a pure strategy is strictly dominated by a mixed strategy, then for every interior initial condition this pure strategy is eliminated by the two-population replicator dynamics.*

Proof. This corresponds to the special case of property 2 where $S_2^+ = S_2$. Since a dynamics cannot eliminate all the pure strategies of a player, the result follows. ■

2.3 Proof of the proposition

We are now ready to prove the proposition. From now on, there are only three pure strategies: $S = \{1, 2, 3\}$, and strategy 3 is not used in correlated equilibrium. The aim is to show that strategy 3 is eliminated by the replicator dynamics. By symmetry, we only need to show that $x_3(t)$ converges to 0. We first exploit the inequations (2) and (3). These inequations are particularly interesting in two cases: first, if $j = i$ then (2) yields

$$U(\alpha * i, i) \geq U(i, i) \tag{4}$$

If moreover strategy i is not used in correlated equilibrium then (3) yields:

$$U(\alpha * i, i) > U(i, i)$$

In particular,

$$U(\alpha * 3, 3) > U(3, 3) \tag{5}$$

Second, if j is α-invariant, i.e. if $\alpha * j = j$, then (2) yields

$$U(\alpha * i, j) \geq U(i, j) \tag{6}$$

If moreover strategy i is not used in correlated equilibrium then (3) yields:

$$U(\alpha * i, j) > U(i, j) \tag{7}$$

Now, distinguish the following cases:

Case 1 *If one of the strategies 1 and 2 is α-invariant*

Assume, for instance, that strategy 1 is α-invariant. Then, by (7),

$$U(\alpha * 3, 1) > U(3, 1) \tag{8}$$

and by (6)

$$U(\alpha * 2, 1) \geq U(2, 1) \tag{9}$$

Furthermore, taking $i = 3$ and $j = 2$ in (3) yields:

$$[U(\alpha * 3, 2) - U(3, 2)] + [U(\alpha * 2, 3) - U(2, 3)] > 0$$

Thus, at least one of the two brackets must be positive. If the first bracket is positive (subcase 1.1), i.e. if $U(\alpha * 3, 2) > U(3, 2)$, then (5) and (8) imply that $\alpha * 3$ strictly dominates 3, hence $x_3(t)$ converges to 0 by property 3 and we are done.

Otherwise (subcase 1.2), the second bracket is positive, i.e. $U(\alpha * 2, 3) > U(2, 3)$. Together with (4) and (9) this implies that $\alpha * 2$ weakly dominates strategy 2, with strict inequality against strategy 3. Thus, by property 2, if $x_3(t)$ does not converge to 0, then $y_2(t)$ does. Now consider the 3×2 game obtained by eliminating the second strategy of player 2: in this reduced game, by (5) and (8), the third strategy of player 1 is strictly dominated by $\alpha * 3$; furthermore, since strategy 2 is weakly dominated, it follows that, in the reduced game, strategy 3 is strictly dominated by strategy 1. This implies that there exists a positive ϵ such that, once $y_2(t)$ is low enough, $U(3, \mathbf{y}) \leq U(1, \mathbf{y}) - \epsilon$. By property 1, this implies that $x_3(t)$ converges to 0.

Case 2 *If neither strategy* 1 *nor strategy* 2 *is invariant*

Consider the 2×2 game G_r obtained by elimination of the third strategy of both players. Since G_r is a 2×2 symmetric game, it may a priori be of three kinds:

Subcase 2.1 a coordination game, i.e. a game with two strict Nash equilibria and a completely mixed one

Subcase 2.2 a game with a weakly or strictly dominated strategy

Subcase 2.3 a trivial game, i.e. a game where the players have no influence on their own payoff.

Since α is a strong dual vector and since strategy 3 is not used in correlated equilibrium, it follows from (Viossat, 2004a, proof of proposition 5.16, steps 2 and 3) that strategy 3 is transient under the Markov chain on S induced by α. This implies that the support of $\alpha * 1$ (resp. $\alpha * 2$) contains strategy 2 (resp. 1) but not strategy 3. This, in turn, implies two things:

First, the game G_r has no strict Nash equilibrium (indeed, if (i', j') is a strict Nash equilibrium of G_r then the inequality (2) for $i = i'$ and $j = j'$ cannot be satisfied). This rules out subcase 2.1.

Second, the Markov chain on S induced by α has a unique recurrent communicating set: $\{1, 2\}$. By the basic theory of dual reduction (Myerson, 1997), this implies that the game G may be reduced, in the sense of dual reduction, into a game with a unique strategy profile, which corresponds to a mixed strategy profile of G with support $\{1,2\} \times \{1,2\}$. By proposition 5.9 of (Viossat, 2004a), this implies that G has a Nash equilibrium with support $\{1,2\} \times \{1,2\}$ and so, that G_r has a completely mixed Nash equilibrium. This rules out subcase 2.2. Thus, the game G_r is necessarily a trivial game.

Now, only two possibilities remain: first (subcase 2.3.1), it may be that $U(1,3) = U(2,3)$, so that $U(1,j) = U(2,j)$ for all j in S. In that case, $U(\alpha * i, j) = U(i,j)$ for every i in $\{1,2\}$ and every j in S (recall that for i in $\{1,2\}$, $\alpha * i$ has support in $\{1,2\}$). Therefore, repeated applications of (3) show that strategy 3 is strictly dominated by $\alpha * 3$, so that $x_3(t)$ converges to 0.

Otherwise (subcase 2.3.2), $U(1,3) \neq U(2,3)$, so that we may assume for instance $U(1,3) > U(2,3)$. This implies that strategy 2 is weakly dominated by strategy 1, with strict inequality against strategy 3. Thus, if $x_3(t)$ does not converge to 0, then $y_2(t)$ does. But in the 3×2 game obtained by elimination of the second strategy of player 2, strategy 1 strictly dominates 3. Therefore, as in subcase 1.2, $x_3(t)$ converges to 0.

3 Extensions and comments

3.1 Other dynamics

The only properties of the replicator dynamics that are used in the proof of the proposition are properties 1 to 3 of section 2.2. Thus, the proposition extends to any dynamics that satisfies these properties. This is the case in particular of the *best-response dynamics* of Matsui (1992) and of the *convex monotonic dynamics* of Hofbauer and Weibull (1996). See the appendix.

3.2 Asymmetric games

When considering multi-population dynamics, there is no compelling reason to focus on symmetric games. A more general result than our proposition would consist in proving that the two-population replicator dynamics eliminates all strategies that are not played in correlated equilibrium in every 3×3 game (and not only in symmetric ones).[1]. I do not know whether this is true or not. However, it may be shown that in every 2×2 game, the two-population replicator dynamics eliminates all strategy *profiles* that are not used in Nash equilibrium in the following sense: if the pure strategy profile (i,j) has probability zero in all Nash equilibria, then $x_i(t)y_j(t) \to 0$ as $t \to +\infty$ for any interior initial condition $(\mathbf{x}(0), \mathbf{y}(0))$. This implies that in every 2×2 game, the two-population replicator dynamics eliminates all strategies that are not used in Nash equilibrium.

Here again, the proof relies solely on properties 1 to 3, so that this extends to the two-population best-response dynamics as well as to any two-population convex monotonic dynamics.

[1]For two-player nonsymmetric games, the replicator dynamics is defined by taking $\mathbf{x}$ in $\Delta(S_1)$, $\mathbf{y}$ in $\Delta(S_2)$ and by replacing U by U_1 (resp. U_2) in the first (resp. second) equation of (1)

3.3 Nash Equilibrium

Zeeman (1980, p.488-489) studies the behaviour of the single population replicator dynamics in 3×3 symmetric games. In all the cases he considered, every strategy that is not used in *symmetric Nash* equilibrium is eliminated by the single population replicator dynamics, for any interior initial condition. This suggests that:

Conjecture *In any 3×3 symmetric game, every strategy that is not used in* symmetric Nash *equilibrium is eliminated by the single population replicator dynamics, for any interior initial condition.*

This result would be stronger than our corollary. Its proof however (if in the spirit of Zeeman (1980)) would probably be much more involved, and maybe less amenable to extensions to other dynamics and to the two-population replicator dynamics.

3.4 Higher dimensional games

It seems that the result of this paper is one of the many results of game theory which holds true only in small dimensions. Indeed, in 4×4 symmetric games (and in higher dimensions), the single population replicator dynamics need not eliminate the pure strategies that are not played in correlated equilibrium. Actually, for the replicator and the best-response dynamics, as well as for every smooth payoff monotonic dynamics (for a definition of payoff monotonic dynamics, see Hofbauer and Sigmund, 1998, p. 88), there exists a 4×4 symmetric game in which, for an open set of initial conditions, all strategies that are used in correlated equilibrium are eliminated (hence, in the long run, only strategies that are NOT used in correlated equilibrium remain). See (Viossat, 2004b) for a proof and related results.

A Some properties of convex monotonic dynamics and of the best-response dynamics

The aim of this section is to show that every convex monotonic dynamics (in particular, the replicator dynamics), as well as the best-response dynamics, checks properties 1 to 3 of section 2.2. We actually prove a slightly more general result.

Consider a two-player game with pure strategy set S_k and payoff function U_k for player k. Consider a dynamics of the form

$$\dot{x}_i = x_i g_i(\mathbf{x}, \mathbf{y}) \qquad \dot{y}_j = y_j h_j(\mathbf{y}, \mathbf{x}) \tag{10}$$

where the C^1 functions g_i (resp. h_j) have the property that $\sum_{i \in S_1} x_i g_i(\mathbf{x}, \mathbf{y}) = 0$ (resp. $\sum_{j \in S_2} y_j h_j(\mathbf{y}, \mathbf{x}) = 0$) for all $(\mathbf{x}, \mathbf{y})$ in $\Delta(S_1) \times \Delta(S_2)$, so that $\Delta(S_1) \times \Delta(S_2)$ and its boundary faces are invariant. The replicator dynamics corresponds to the special case $g_i(\mathbf{x}, \mathbf{y}) = U_1(i, \mathbf{y}) - U_1(\mathbf{x}, \mathbf{y})$ and $h_j(\mathbf{y}, \mathbf{x}) = U_2(j, \mathbf{x}) - U_2(\mathbf{y}, \mathbf{x})$.

Such a dynamics (10) is *convex monotonic* if

$$U_1(\mathbf{p}, \mathbf{y}) > U_1(i, \mathbf{y}) \Rightarrow \sum_{k \in S_1} p_k g_k(\mathbf{x}, \mathbf{y}) > g_i(\mathbf{x}, \mathbf{y})$$

for all i in S_1, all **p** in $\Delta(S_1)$, and all $(\mathbf{x}, \mathbf{y})$ in $\Delta(S_1) \times \Delta(S_2)$, and similarly

$$U_2(\mathbf{q}, \mathbf{x}) > U_2(j, \mathbf{x}) \Rightarrow \sum_{k \in S_2} q_k h_k(\mathbf{y}, \mathbf{x}) > h_j(\mathbf{y}, \mathbf{x})$$

for all j in S_2, all **q** in $\Delta(S_2)$, and all $(\mathbf{y}, \mathbf{x})$ in $\Delta(S_2) \times \Delta(S_1)$. In particular, the replicator dynamics is convex monotonic.

Let $i \in S_1$ and $\mathbf{p} \in \Delta(S_1)$ denote respectively a pure and a mixed strategy of player 1. Fix a convex monotonic dynamics and an interior initial condition $(\mathbf{x}(0), \mathbf{y}(0))$.

Proposition Assume that at least one of the following conditions holds:

1. There exist $\epsilon > 0$ and a time T in $\mathbb{R}$ such that, for all $t \geq T$, $U_1(i, \mathbf{y}(t)) \leq U_1(\mathbf{p}, \mathbf{y}(t)) - \epsilon$
2. The mixed strategy **p** weakly dominates i and there exists a pure strategy j in $S_2^+ = \{j \in S_2, U_1(\mathbf{p}, j) > U_1(i, j)\}$ such that $y_j(t)$ does not go to zero as time goes to infinity.

Then $x_i(t)$ converges to 0 as time goes to infinity.

Proof. The below argument is essentially the one used by Hofbauer and Weibull (1996) to show that convex monotonic dynamics eliminate the pure strategies that are iteratively strictly dominated. Let $V(\mathbf{x}) := x_i \prod_{k \in S_1} x_k^{-p_k}$ and $W(t) = V(\mathbf{x}(t))$. We have:

$$\dot{W}(t) = \sum \frac{\partial V(\mathbf{x})}{\partial x_j} \dot{x}_j = W(t) \left(g_i(\mathbf{x}, \mathbf{y}) - \sum p_k g_k(\mathbf{x}, \mathbf{y}) \right)$$

If condition 1 holds, then for all time $t \geq T$, $\mathbf{y}(t)$ belongs to the compact set

$$K_\epsilon = \{\mathbf{y} \in \Delta(S_2), U_1(\mathbf{p}, \mathbf{y}) \geq U_1(i, \mathbf{y}) + \epsilon\}$$

Since for y in K_ϵ, the quantity where $g_i(\mathbf{x}, \mathbf{y}) - \sum p_k g_k(\mathbf{x}, \mathbf{y})$ is always positive, hence, by continuity, always greater that some positive constant α, it follows that, for $t \geq T$, $\dot{W} \leq \alpha W$. Therefore,

$$t \geq T \Rightarrow W(t) \leq W(T) \exp(\alpha(t - T))$$

This implies that $x_i(t)$, which is smaller than $W(t)$, converges to 0.

If condition 2 holds, then for all time $t \geq 0$, $g_i(\mathbf{x}, \mathbf{y}) - \sum p_j g_j(\mathbf{x}, \mathbf{y})$ is negative, hence W is always decreasing. Furthermore, there exists $\epsilon \geq 0$ and $j \in S_2^+$ such that $\limsup y_j(t) \geq \epsilon$. Since the velocity $\dot{y}_j$ is bounded, it follows that $\mathbf{y}(t)$ spends an infinite amount of time within the compact set

$$K'_\epsilon = \{\mathbf{y} \in \Delta(S_2), y_j \geq \epsilon/2\}$$

Let $\tau(t)$ be the time spent by **y** in K'_ϵ up to time t:

$$\tau(t) = \int_0^t \mathbf{1}_{y_j \geq \epsilon/2}(\mathbf{y}(t))\, dt$$

(where $\mathbf{1}_{y_j \geq \epsilon/2}(\mathbf{y})$ is equal to 1 if $y_j \geq \epsilon/2$ and to 0 otherwise).

Since for $\mathbf{y}$ in K'_ϵ, the quantity $g_i(\mathbf{x}, \mathbf{y}) - \sum p_j g_j(\mathbf{x}, \mathbf{y})$ is always positive, hence, by continuity, always greater that some positive constant α', it follows that:

$$x_i(t) \leq W(t) \leq W(0) \exp(-\alpha' \tau)$$

Since $\tau(t)$ goes to infinity, this implies that $x_i(t)$ converges to 0. ■

Now consider the best-response dynamics (Matsui, 1992) given by:

$$\dot{\mathbf{x}} \in BR(\mathbf{y}) \text{ and } \dot{\mathbf{y}} \in BR(\mathbf{x})$$

where $BR(\mathbf{y}) \subseteq \Delta(S_1)$ (resp. $BR(\mathbf{x}) \subseteq \Delta(S_2)$) is the of set of mixed best-responses to $\mathbf{y}$ (resp $\mathbf{x}$):

$$BR(\mathbf{y}) = \{\mathbf{p} \in \Delta(S_1), \forall i \in S_1, U_1(\mathbf{p}, \mathbf{y}) \geq U_1(i, \mathbf{y})\}$$

Fix an arbitrary initial condition $(\mathbf{x}(0), \mathbf{y}(0))$. Assuming that one of the two conditions of the above proposition holds, then, at least after some time T, the pure strategy i is not a best-response to $\mathbf{y}(t)$ (for the second condition, use the fact that under the best-response dynamics: if $y_j(t) > 0$, then $y_j(t') > 0$ for all $t \geq t'$). Therefore, $\mathbf{x}_i(t)$ decreases exponentially to zero. Thus, the above proposition extends to the best-response dynamics.

This implies that the best-response dynamics, as well as any convex monotonic dynamics, checks properties 1 to 3 of section 2.2.

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