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PRICING IN NETWORKS

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Résumé: Cet article analyse la fixation des prix dans un réseau en présence d'externalités de prix ou de consommation. Nous étudions la relation entre les prix et des mesures de centralité des points du réseau. En utilisant une approche asymptotique, nous montrons que l'ordre des prix et des stratégies optimales peut être approximé par un ordre lexicographique sur des caractéristiques nodales du réseau. Nous montrons que quand les agents font face à des externalités de consommation positives, les prix sont plus élevés dans les points du réseau au degré plus élevé. Quand les agents font face à des externalités relatives de prix, les prix sont plus élevés aux points du réseau qui ont le plus de voisins à faible degré.

Abstract: This paper studies optimal pricing in networks in the presence of local consumption or price externalities. It analyzes the relation between prices and nodal centrality measures. Using an asymptotic approach, it shows that the ranking of optimal prices and strategies can be reduced to the lexicographic ranking of a specific vector of nodal characteristics. In particular, this result shows that with positive consumption externalities, prices are higher at nodes with higher degree, and with relative price externalities, prices are higher at nodes which have more neighbors of smaller degree.

Mots clés : Réseaux sociaux, externalités de réseau, oligopoles

Key Words : Social Networks, Network Externalities, Oligopolies

Classification JEL: D85, D43, C69

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1 Introduction

Consumption behavior is often affected by social interactions. Consumers share information about the price and quality of products, compare their consumption of status goods with that of their neighbors, benefit from network externalities by consuming the same goods as members of their professional or social circles.[1] In many instances, firms are aware of the influence of social relations on the consumption of the products they sell, and try to exploit the underlying network externalities to maximize profit. To this end, they may charge different prices at different nodes of the social network. In this paper, our objective is precisely to analyze optimal discriminatory pricing strategies in the presence of local network externalities.

Our analysis focusses around two questions. First, we investigate how prices reflect characteristics of the nodes in the social network. In particular, we study how classical network centrality measures (like degree centrality or eigenvector centrality) are related to prices.[2] Second, we study how changes in social network affect the prices charged by the firms. More specifically, we analyze the impact of the addition of a new link on the prices charged at neighboring nodes.

These two types of comparative statics exercices have recently attracted considerable attention from economists studying the effects of social networks on economic activities.[3] In a remarkable contribution, Ballester, Calvó-Armengol and Zenou (2006) have proposed a method to analyze these two questions in the context of linear-quadratic games, where agents' objective functions are quadratic, and interior equilibria can be computed as solutions to a system of linear equations. They exhibit a relation between an agent's optimal decision and the Bonacich (1987) measure of network centrality, which computes the discounted sums of paths originating from an agent in the network.[4]

In line with the general approach of Ballester, Calvó-Armengol and Zenou (2006), we model pricing in a network as a linear-quadratic problem, by

[1]Typical examples of local network externalities are the use of common software with colleagues or co-authors, the purchase of books or movies recommended by friends, sensitivity to fashion or snob effects, etc..

[2]See Wasserman and Faust (1994) for a clear exposition of the literature on network centrality measures.

[3]For excellent recent surveys on the economic literature on networks, see the books by Sanjeev Goyal (2007) and Matt Jackson (2008).

[4]In the very different context of large networks, which are characterized by their degree distribution, Sundarajan (2006), Galeotti et al. (2006) and Galeotti and Goyal (2007) have studied the relation between optimal economic decisions and agents' degree centrality.

assuming that consumer's utilities are quadratic in consumption, so that demand functions are linear.[5] We model the impact of social interactions on consumption in different ways. We first consider a model where players care about the level of consumption of other consumers in their neighborhood. Consumption externalities can either be positive, as in the case of local network externalities generated by the use of common software or common products, or negative as in the case of consumption of status goods. We also consider a model where agents care about the prices charged to their neighbors. Two different models of price externalities are analyzed: one where consumers care about the average price charged in their neighborhood, and one where utilities are affected by the sum of prices charged to a consumer's neighbors.

The study of these different models of pricing with social interactions highlights the power and the limits of the general approach of Ballester, Calvó-Armengol and Zenou (2006). On the one hand, the methodology they propose delivers exact results on the range of parameters for which unique interior solutions exist, and provides an exact formula to compute the equilibrium. On the other hand, in models where firms' objective functions involve complex transformations of the matrix of social interactions, the Bonacich (1987) centrality measure loses its transparent interpretation. Hence, the exact result of Ballester, Calvó-Armengol and Zenou (2006) cannot be used to relate prices charged at different nodes of the social network to simple characteristics of the nodes.

In order to investigate further the relation between nodal characteristics and firms' prices, we propose an asymptotic approach when the magnitude of externalities converges to zero. More precisely, we analyze solutions of systems of linear equations where the coefficients of the matrix can be written as power series in a given parameter λ , such that the off-diagonal terms of the matrix converge to zero when λ goes to zero. Using standard techniques of matrix norms, we show that the solution to the system of linear equations can itself be written as a power series in λ. Hence, in order to compare solutions of the systems of equations when λ goes to zero, one only needs to consider the lexicographic ordering of the coefficients of the power series. This approach allows us to rank equilibrium prices as a function of simple nodal characteristics, like the degree and the sum of neighbors' degrees.

At the outset, we would like to defend the usefulness of computing ap-

[5] The use of quadratic utility functions in order to generate linear demand functions is standard in oligopoly theory. An early application of quadratic utility functions in Industrial Organization can be found in Shubik and Levitan (1980)'s classic book on oligopoly theory.

proximation results when the magnitude of local external effects goes to zero. First, as Ballester, Calvo-Armengol and Zenou (2006) argue, the existence of a unique interior equilibrium is guaranteed if and only if the magnitude of local effects is small – the largest eigenvalue of the matrix of local effects must be bounded above by one.[6] Hence, analyzing a model with small local effects fits well with their approach. Second, when the matrix of external effects is complex, the asymptotic approach may be the only way to handle a problem which would otherwise be intractable. Third, by continuity, the intuition obtained for small external effects continues to hold when the magnitude of externalities increases, so that our qualitative results remain true for a wider range of situations.

In a model with positive consumption externalities, the asymptotic approach shows that competing firms charge higher prices at nodes with higher degree. If two nodes have the same degree, prices are higher at the node for which the sum of neighbors' degree is highest. Furthermore, the addition of a new link between i and j increases the prices charged at i and j and all their neighbors. These results do not hold when a single firm serves the entire network. The monopolist charges uniform prices, consumers at nodes with higher Bonacich centrality consume more, and obtain higher utility. When consumption externalities are negative, the results are reversed. Prices set by competing firms are lower at nodes with higher degree, and while the monopolist charges uniform prices at every node, consumers at node with higher centrality consume less and obtain lower utility.

To understand the intuition underlying these comparisons, notice that in models with positive consumption externalities, consumers at nodes with higher degree have higher demand. Hence firms serving nodes with higher degree typically have higher best-response functions, resulting in higher equilibrium prices. When a single firm serves all markets, it faces a new trade-off. By increasing the price at nodes with higher degree, it reduces demand at all the neighbors' nodes. In the linear model we analyze, the incentive to increase and lower prices at nodes with higher degree are exactly balanced, and the monopolist charges uniform prices on the network.

In a model with average price externalities where consumer's utility is increasing in the average price charged to neighbors, competing firms charge uniform prices in the social network. The social externality vanishes at equi-

[6]This distinguishes their analysis from Bramoullé and Kranton (2007)'s model, where local effects are not small, and multiple corner equilibria are present. See also Bramoullé, Kranton and d'Amours (2008) for a model which reconciles the two approaches and Ballester and Calvó-Armengol (2007) for more precise results and tighter bounds on existence and uniqueness of equilibrium.

librium, but prices remain higher than they would be in the absence of externalities. When a single firm serves all nodes, it charges differentiated prices. The asymptotic approach shows that prices are higher when the sum of the inverse of the degrees of the neighbors is larger. The monopolist thus has an incentive to raise the price at nodes which have a large influence on their neighbors, i.e. nodes which have many neighbors with small degree. For example, in a star network, the monopolist has a strong incentive to raise the price of the hub of the star, in order to increase demand at the peripheral nodes.

Finally, when consumers care about the sum of prices charged in their neighborhood, prices are proportional to the Bonacich centrality measure of the nodes in the social network. Consumers at nodes with higher centrality will experience higher prices, both when different firms compete in the network and when a single firm serves all nodes.

We now comment on the relation between our paper and other recent work on consumption externalities in networks. Galeotti (2006) studies information sharing among consumers in a model of search. Jullien (2001), Sundarajan (2006) and Banerji and Dutta (2005) analyze models with positive local network externalities. Jullien (2001) and Banerji and Dutta (2005) focus on competition among two price-setting firms. While Banerjee and Dutta (2006) consider uniform prices, Jullien (2001) allows for discriminatory pricing at different nodes, and provides partial results suggesting that firms set lower prices at nodes with higher degree. Sundarajan (2006) studies monopoly pricing and focusses attention on consumer's adoption decisions. In that sense, his model is closely related to Galeotti et al. (2006)'s study of network games with binary decisions, and both papers show that the decision to buy a new product is increasing in a consumer's degree. This suggests that, as in our paper, demand will be higher at nodes with higher degree. Finally, Ghiglino and Goyal (2008) study a general equilibrium model where consumers care negatively about the consumption of their neighbors. They also adopt a quadratic-linear framework and characterize equilibrium prices as a function of the social network. Prices in their model are determined as part of a competitive equilibrium, while we consider strategic pricing by oligopolistic firms. This difference in settings precludes a direct comparison between their results and ours, but the thrust of the analysis is very similar.

The rest of the paper is organized as follows. In Section 2, we introduce our asymptotic approach. Section 3 discusses the model with consumption externalities. Section 4 is devoted to the model with price externalities. We conclude in the last Section.

2 Mathematical Preliminaries

In this Section, we state two technical results which form the core of the asymptotic approach to comparative statics in networks. These results have no economic content and are direct applications of standard techniques in matrix algebra. We have chosen to present them in a separate section and isolate them from the rest of the paper, because we believe that the asymptotic approach might be useful in a broad array of applications.

Consider an abstract system of linear equations

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{a}. \tag{1}$$

where $\mathbf{A}$ is a $n \times n$ nonnegative square matrix, and $\mathbf{a}$ a positive vector in $\Re^n$. Suppose furthermore that there exist sequences of nonnegative square matrices $(\mathbf{A}_1, ..., \mathbf{A}_l, ...)$ (which can be equal to the zero matrix for some ls), and a sequence of nonnegative vectors $(\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_l, ...)$ (which can be equal to zero for some $l > 1$) such that

$$\mathbf{A} = \sum_{l=1}^{\infty} \lambda^l \mathbf{A}_l, \tag{2}$$

$$\mathbf{a} = \sum_{l=0}^{\infty} \lambda^l \mathbf{a}_l. \tag{3}$$

In words, we consider systems of linear equations which are parametrized by a positive scalar λ, and such that every coefficient of the system of equations can be written as a power series in λ. Furthermore, as λ goes to zero, the off-diagonal terms of the matrix $\mathbf{I} - \mathbf{A}$ converge to zero. We will investigate properties of the solutions to the system of linear equations when λ becomes small.

First observe that, when $\lambda = 0$, the system of equations admits a unique solution

$$\mathbf{x} = \mathbf{a}_0 \tag{4}$$

By continuity, there exists $\overline{\lambda} > 0$ such that, for all $\lambda < \overline{\lambda}$, the system of equations admits a unique interior solution given by

$$
\begin{aligned}
\mathbf{x} &= (\mathbf{I}-\mathbf{A})^{-1}\mathbf{a}, && (5)\\
&= \sum_{k=0}^{\infty}\mathbf{A}^k\mathbf{a}, && (6)\\
&= \sum_{k=0}^{\infty}(\sum_{l=1}^{\infty}\lambda^l\mathbf{A}_l)^k(\sum_{l=0}^{\infty}\lambda^l\mathbf{a}_l). && (7)
\end{aligned}
$$

We now need to express the vector $\mathbf{x}$ as a power series in λ,

$$\mathbf{x} = \sum_{k=0}^{\infty}\lambda^k\mathbf{c}^k. \quad (8)$$

To this end, we will use a variant of the Faà di Bruno formula on the composition of abstract power series to compute the vectors $\mathbf{c}^k$.[7]

Definition 2.1 *A partition of an integer m, $p(m)$ is a sequence of positive integers, $(p_1, ..., p_R)$ such that $\sum_r p_r = m$. The set of all partitions of an integer m is denoted $\mathcal{P}(m)$. By convention, suppose that the partition of 0 is 0 and let $\mathbf{I}$ be the matrix corresponding to $\mathbf{A}_0$.*

The sequence of vectors $\mathbf{c}_k$ is given by:

$$\mathbf{c}^k = \sum_{t=0}^{k}\sum_{(p_r)\in\mathcal{P}(k-t)}\prod_r \mathbf{A}_{p_r}\mathbf{a}_t. \quad (9)$$

Arguably, this sequence is not easy to compute. However, the first terms are rather straightforward and given by:

$$
\begin{aligned}
\mathbf{c}^0 &= \mathbf{a}_0,\\
\mathbf{c}^1 &= \mathbf{a}_1 + \mathbf{A}_1\mathbf{a}_0,\\
\mathbf{c}^2 &= \mathbf{a}_2 + \mathbf{A}_1\mathbf{a}_1 + (\mathbf{A}_1^2 + \mathbf{A}_2)\mathbf{a}_0,\\
\mathbf{c}^3 &= \mathbf{a}_3 + \mathbf{A}_1\mathbf{a}_2 + (\mathbf{A}_1^2 + \mathbf{A}_2)\mathbf{a}_1 + (\mathbf{A}_1^3 + \mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_3)\mathbf{a}_0.\\
&\dots
\end{aligned}
$$

Next, recall the definition of the lexicographic ordering of two sequences:

[7] For a presentation of Faà di Bruno's formula and its variants, see Johnson (2002)'s historical article and the references therein.

Definition 2.2 *The sequence* $f = (f_1, ..., f_k, ...)$ lexicographically dominates *the sequence* $f' = (f'_1, ..., f'_k)$, $f \prec f'$ *if and only if there exists* K *such that* $f_k = f'_k$ *for all* $k < K$ *and* $f_K > f'_K$.

We are now ready to prove the approximation lemma, which provides an equivalence between the ranking of the components of the solution $\mathbf{x}$ and the lexicographic ordering of the components of the sequence $(\mathbf{c}^0, \mathbf{c}^1, ...)$ when λ converges to zero.

Lemma 2.3 *Consider a sequence of positive scalars* λ_t *converging mononotically to zero. There exists* $T > 0$ *such that, for all* $t > T$, *the system of linear equations (1) has a unique interior, positive solution* $\mathbf{x}^t$ *and for any* i, j,

$$x_i^t > x_j^t \Leftrightarrow (\mathbf{c}_i^0, \mathbf{c}_i^1, ...) \succ (\mathbf{c}_j^0, \mathbf{c}_j^1, ...)$$

Proof. See the Appendix. ■

Lemma 2.3 shows that, in order to compare two different components of the vector of solutions $\mathbf{x}$, we can restrict attention to the zero order term $\mathbf{c}^0$, or if the zero order terms are equal, to the first order term $\mathbf{c}^1$, and if the first order terms are equal to the second order term $\mathbf{c}^2$, etc.. The intuition underlying the result is easily grasped. We can use the composition of formal power series to write down the solution $\mathbf{x}$ as a power series in λ. When λ converges to zero, higher order terms become negligible, and the comparison between two components of the solution vector $\mathbf{x}$ only depend on the ranking of lower order terms.[8]

The same approximation can also be used to compare the solutions to two different systems of equations.

Lemma 2.4 *Consider two systems of equations* $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{a}$ *and* $(\mathbf{I} - \mathbf{A}')\mathbf{x} = \mathbf{a}'$. *Let* $(\mathbf{c}^0, \mathbf{c}^1, ...)$ *and* $(\mathbf{c}'^0, \mathbf{c}'^1, ...)$ *denote the corresponding sequences. Consider a sequence of positive scalars* λ_t *converging monotonically to zero. There exists* $T > 0$ *such that, for all* $t > T$, *the two systems of linear equations have unique, interior positive solutions* $\mathbf{x}^t$ *and* $\mathbf{x}'^t$ *and for any* i,

$$x_i^t > x_j'^t \Leftrightarrow (\mathbf{c}_i^0, \mathbf{c}_i^1, ...) \succ (\mathbf{c}_i'^0, \mathbf{c}_i'^1, ...)$$

[8]This intuition is almost entirely correct. The only remaining step is to show that, not only do the higher terms become negligible, but that the *sum* of higher terms also becomes small. In other words, we need to show that the series of higher order terms converges.

Proof. See the Appendix. ■

Lemma 2.4 is a useful tool to study the effects of changes in the social network on the solutions $\mathbf{x}$. Changes in the social network typically will result in changes in the matrix $\mathbf{A}$ and the vector $\mathbf{a}$, and in many instances, it will be rather straightforward to identify the first term of the sequence $\mathbf{c}_i$ which is affected by the change in the social network.

3 Pricing with Consumption Externalities

3.1 The Model

3.1.1 Consumer's utilities and choices

In this Section, we consider a model where consumers' utilities are affected by the *quantities* consumed by other consumers on the network. We suppose that consumers are located on a social network, defined by an undirected graph g with n nodes indexed by $i = 1, 2, ..., n$. The matrix $\mathbf{G}$ denotes the adjacency matrix of graph g, with typical entry $g_{ij} \in \{0, 1\}$. The matrix $\overline{\mathbf{G}}$ denotes the complementary of the adjacency matrix, namely a matrix such that $\overline{g_{ij}} = 1$ if and only if $g_{ij} = 0$. Correspondingly, we let $\overline{g}$ denote the complementary of network g. We let $\deg i$ denote the *degree* of node i, or the number of links originating at i,

$$\deg i = \sum_j g_{ij}.$$

The vector $\mathbf{d} = (d_1, ..., d_n)$ collects the degrees of all nodes in the network, whereas the vector $\mathbf{id} = (\frac{1}{d1}, ..., \frac{1}{d_n})$ collects the inverses of the degrees of all nodes in the network.

At each node i of the network, consumer i's utility is defined over her consumption, q_i, the consumption of her neighbors $\sum_j g_{ij} q_j$, and the price at node i, p_i. We suppose that the utility is linear-quadratic and given by:

$$U_i = q_i - \frac{1}{2} q_i^2 + \lambda \sum_j g_{ij} q_i q_j - p_i q_i. \tag{10}$$

Note that in this formulation λ can either be positive or negative. If $\lambda > 0$, our model is a model of *local positive network externalities* where consumers benefit from the consumption of the good by their neighbors. This is the typical case of network externalities studied by Farrell and Saloner (1985) and Katz and Shapiro (1985). If $\lambda < 0$, our model is a model of *negative consumption externalities*, where consumers are harmed by the consumption of their neighbors. This is the classical model of conspicuous or status goods first emphasized by Veblen (1899), and recently studied in the context of social networks by Ghiglino and Goyal (2008). Consumers compare their consumption of a status good with that of their neighbors and derive positive utility from consuming more than their neighbors. We will furthermore assume that $\lambda > -1$, which implies that own consumption has higher weight than the social externality in every consumer's utility.

Alternative Interpretations Alternatively, we could identify nodes of the network g as geographical locations, and have a continuum of consumers of measure 1 at each node. The network g is then interpreted as a transportation network among geographical locations. Assume that consumers can only buy the good at the location where they reside (either because of transportation costs or specific regulations) but derive positive or negative utility from consumption in neighboring locations.

As another interpretation, we could identify nodes of the network g as variants of a product. If $\lambda > 0$, the network g measures the complementarity of the products: if $g_{ij} = 1$, products i and j are complementary, if $g_{ij} = 0$, they are independent. If $\lambda < 0$, the network g measures the substitutability of the products: if $g_{ij} = 1$, products i and j are substitutable, if $g_{ij} = 0$, they are independent. Notice that we need not assume that complementarity or substitutability are transitive: consumers can view goods i and j and j and k as complements, but not perceive any relation between products i and k. Similarly, goods i and j and j and k may be substitutable, but goods i and k display no substitutability. We could also interpret the network of products in terms of technical compatibility rather than consumers' perceptions. Two products for which $g_{ij} = 0$ are incompatible, two products for which $g_{ij} = 1$ are compatible (and can either be substitutes if $\lambda < 0$ or complements if $\lambda > 0$). Again, we do not need to assume that compatibility is transitive. With this interpretation in mind, we do not associate consumers to nodes and assume instead that there is a continuum of identical consumers, with a utility defined over the variants of all products,

$$U_i = \sum_i q_i - \frac{1}{2}\sum_i q_i^2 + \lambda \sum_i \sum_j g_{ij} q_i q_j - \sum_i p_i q_i. \tag{11}$$

This model gives rise to a demand system which is slightly different from that of Equation (10). However, the basic results of our analysis extend to this alternative model.

3.1.2 Optimal consumer choices

Given any vector of prices $\mathbf{p} = (p_1, p_2, ..., p_n)$, consumers choose the quantities in order to maximize the utility given in Equation (10). Given the interdependence arising from consumption externalities, the optimal choice of consumers are given by the Nash equilibrium of a noncooperative game played by consumers at different nodes of the social network. We apply directly the results of Ballester, Calvó-Armengol and Zenou (2006) on the equilibrium of a game played by agents on a social network with linear-quadratic objective functions.

We first recall the definition of the Bonacich centrality measure of nodes in a social network. For any nonnegative scalar $a \geq 0$ and square matrix $\mathbf{G}$, define the matrix

$$\mathbf{M}(\mathbf{G}, a) = [\mathbf{I} - a\mathbf{G}]{-1} = \sum_{k=0}^{\infty} a^k \mathbf{G}^k.$$

Let $\mathbf{1}$ denote the vector of 1's.

Definition 3.1 *Let* $\mathbf{G}$ *be the adjacency matrix of a network* g *and* a *a scalar such that* $\mathbf{M}(\mathbf{G}, a)$ *is well defined and nonnegative. The vector of* Bonacich centrality measures *in network* g *with scalar* a *is given by* $\mathbf{b}(a, g) = \mathbf{M}(\mathbf{G}, a)\mathbf{1}$.

For any agent i, the Bonacich centrality measure $b_i(a, g)$ computes the sum of discounted paths of length k originating at i. For any vector α in $\Re^n$, we also define the *weighted Bonacich centrality measures* as $\mathbf{b}_\alpha(a, g) = \mathbf{M}(\mathbf{G}, a)\alpha$. We also define $b_\alpha(a, g)$ to be the sum of weighted Bonacich centrality measures of all agents in the network, $b_\alpha(a, g) = \mathbf{b}_\alpha(a, g)\mathbf{1}$. We are now ready to apply Theorem 1 in Ballester, Calvó-Armengol and Zenou (2006) to show:

Proposition 3.2 *Consider* $\lambda > 0$ *and suppose that* $\lambda < \frac{1}{\mu_1(\mathbf{G})}$, *where* $\mu_1(\mathbf{G})$ *denotes the largest eigenvalue of the adjacency matrix* $\mathbf{G}$. *Then, for any vector of prices* $\mathbf{p}$, *the game played by consumers at different nodes has a unique interior equilibrium given by*

$$\mathbf{q} = \mathbf{b}_{(\mathbf{1}-\mathbf{p})}(g, \lambda).$$

Consider $-1 < \lambda < 0$ *and suppose that* $-\frac{\lambda}{1+\lambda} < \frac{1}{\mu_1(\mathbf{G})}$. *Then, for any vector of prices* $\mathbf{p}$, *the game played by consumers at different nodes has a unique interior equilibrium given by*

$$\mathbf{q} = \frac{\mathbf{b}_{(\mathbf{1}-\mathbf{p})}(\overline{g}, -\frac{\lambda}{1+\lambda})}{1+\lambda-\lambda b(\overline{g}, -\frac{\lambda}{1+\lambda})}.$$

Proof. The result is a direct application of Theorem 1 in Ballester, Calvó-Armengol and Zenou (2006). Using their notations, we decompose the matrix of cross-effects as

$$\Sigma = -\tilde{\beta}\mathbf{I} - \tilde{\gamma}\mathbf{U} + \tilde{\lambda}\mathbf{G}$$

where $\mathbf{U}$ is the matrix of 1s and $\tilde{\alpha}$ denotes the vector of linear effects.

If $\lambda > 0$, we decompose the matrix of cross-effects with $\tilde{\alpha}_i = (1-p_i), \tilde{\lambda} = \lambda, \tilde{\beta} = 1$, $\tilde{\gamma} = 0$. If $-1 < \lambda < 0$, we decompose the matrix of cross effects with $\tilde{\alpha}_i = (1-p_i), \tilde{\lambda} = -\lambda, \tilde{\beta} = 1+\lambda$, $\tilde{\gamma} = -\lambda$ and $g_{ij} = \overline{g_{ij}}$. ■

Proposition 3.2 provides two results: it first shows that when the magnitude of external effects is not too high, the game admits a unique interior equilibrium. Second, it provides explicit formulas to compute equilibrium quantities at every node as a function of weighted Bonacich centrality measures of the nodes in the appropriate networks (the original network g when $\lambda > 0$ and the complementary network $\overline{g}$ when $\lambda < 0$).

3.1.3 Firms

We consider firms which charge discriminatory prices $p_1, p_2, ..., p_n$ at each node in the network. For simplicity, we normalize the constant marginal cost of production at zero, and define the profit of a firm at node i as

$$\Pi_i(\mathbf{p}) = p_i q_i(\mathbf{p}).$$

We will analyze both the case where each node is served by a different firm and the case where all nodes are served by a single, price discriminating, firm.

3.2 Positive Consumption Externalities

When $\lambda > 0$, using Proposition 3.2, we can rewrite consumer demands as:

$$\begin{aligned} q(\mathbf{p}) &= [\mathbf{I} - \lambda\mathbf{G}]^{-1}(\mathbf{1}-\mathbf{p}), \\ &= \mathbf{b}(g,\lambda) - [\mathbf{I} - \lambda\mathbf{G}]^{-1}\mathbf{p}, \\ &= \mathbf{b}(g,\lambda) - \sum_{k=0}^{\infty} \lambda^k \mathbf{G}^k \mathbf{p}. \end{aligned}$$

Let g^k_{ij} denote the ij entry of the matrix $\mathbf{G}^k$. The preceding computation shows that the demand at node i, q_i can be rewritten as:

$$q_i(\mathbf{p}) = \mathbf{b}_i(g,\lambda) - \sum_j \sum_{k=0}^{\infty} \lambda^k g^k_{ij} p_j. \tag{12}$$

Hence, the demand of consumers at node i is a linear function of the prices $p_1, ..., p_n$, that we may rewrite as:

$$q_i(\mathbf{p}) = \mathbf{b}_i(g,\lambda) - \sum_j c_{ij} p_j, \tag{13}$$

with $c_{ij} = \sum_{k=0}^{\infty} \lambda^k g^k_{ij}$.

3.2.1 The limits of the BCAZ decomposition

We first consider the case where each node is served by a different firm. The profit of firm i operating at node i as a linear quadratic function of the prices $p_1, p_2, .., p_n$, with

$$\Pi_i = p_i \mathbf{b}_i(g,\lambda) - c_{ii} p_i^2 - \sum_{j \neq i} c_{ij} p_i p_j. \tag{14}$$

Hence, the noncooperative game played by the n firms is again a linear-quadratic game and can again be analyzed using the decomposition of Ballester, Calvó-Armengol and Zenou (2006). More specifically, define $\tilde{\alpha_i} = \frac{b_i(g,\lambda)}{2c_{ii}}$, $\tilde{c_{ij}} = \frac{c_{ij}}{2c_{ii}}$. Suppose that $\tilde{c_{ij}} < 1$ for all i, j. Let $\tilde{\gamma} = \max_{i \neq j} \tilde{c_{ij}}$, $\tilde{\beta} = 1 - \tilde{\gamma}$, $\tilde{\lambda} = \max_{i \neq j} \tilde{c_{ij}} - \min_{i \neq j} \tilde{c_{ij}}$ and finally, the weighted graph $\tilde{g}$ be defined by the matrix $\tilde{\mathbf{G}}$ with generic term $\tilde{g_{ij}} = \frac{\max_{i \neq j} \tilde{c_{ij}} - \tilde{c_{ij}}}{\max_{i \neq j} \tilde{c_{ij}} - \min_{i \neq j} \tilde{c_{ij}}}$. We then obtain the following characterization of equilibrium prices:

Proposition 3.3 *Consider price competition among n firms located at different nodes of graph g with positive consumption externalities. If $\frac{\tilde{\lambda}}{\tilde{\beta}} < \frac{1}{\mu_1(\tilde{G})}$, the noncooperative pricing game has a unique interior solution*

$$\mathbf{p} = \frac{\mathbf{b}_{\tilde{\alpha}}(\tilde{g}, \frac{\tilde{\lambda}}{\tilde{\beta}})}{\tilde{\beta} + \tilde{\gamma} b_{\tilde{\alpha}}(\tilde{g}, \frac{\tilde{\lambda}}{\tilde{\beta}})}.$$

Proposition 3.3 highlights the power and the limits of the decomposition of Ballester, Calvó-Armengol and Zenou (2006). On the one hand, this is a very general model which provides exact conditions under which the pricing game admits a unique interior equilibrium, and a formula to compute the equilibrium. On the other hand, the variables used in the decomposition are derived from the primitives of the model (the underlying social network g and the magnitude of local effects λ) in an extremely complex way, and the Bonacich centrality measure $\mathbf{b}_{\tilde{\alpha}}(\tilde{g}, \frac{\tilde{\lambda}}{\tilde{\beta}})$ cannot be interpreted as a simple measure of centrality in the underlying network g.

3.2.2 Ranking of prices at different nodes

In order to better understand the relation between equilibrium prices and the underlying social network g, we consider an asymptotic model, letting the magnitude of local effects, λ, converge to zero. We will use Lemma 2.3 to study the ranking of optimal prices chosen by Bertrand competitors on the network. For λ small enough, the pricing game admits a unique interior equilibrium. Recalling the expression of profits in equation (14), and taking derivatives, we derive the system of linear equations

$$2c_{ii}p_i + \sum_j c_{ij}p_j = \mathbf{b}_i(g, \lambda) \tag{15}$$

This system can be rewritten in matrix form as

$$(\mathbf{C} + \Delta(\mathbf{C}))\mathbf{p} = \mathbf{b}(g, \lambda) \tag{16}$$

where $\Delta(\mathbf{C})$ denotes the diagonal matrix formed by picking the diagonal elements of $\mathbf{C}$, i.e. the diagonal matrix such that $d_{ii} = c_{ii}$ and $d_{ij} = 0$ for $i \neq j$. Recalling the definitions of $\mathbf{C}$ and $\mathbf{b}(g, \lambda)$, we obtain:

$$((\mathbf{I} - \lambda\mathbf{G})^{-1} + \Delta((\mathbf{I} - \lambda\mathbf{G})^{-1}))\mathbf{p} = (\mathbf{I} - \lambda\mathbf{G})^{-1}\mathbf{1}. \tag{17}$$

Premultiplying both sides of the equation by $(\mathbf{I} - \lambda\mathbf{G})$ and rearranging terms, we obtain:

$$((\mathbf{I} + (\mathbf{I} - \lambda\mathbf{G})\Delta((\mathbf{I} - \lambda\mathbf{G})^{-1}))\mathbf{p} = \mathbf{1}. \tag{18}$$

or

$$(2\mathbf{I} - (\mathbf{I} - (\mathbf{I} - \lambda\mathbf{G})\Delta((\mathbf{I} - \lambda\mathbf{G})^{-1})))\mathbf{p} = \mathbf{1}, \tag{19}$$

Dividing by 2, we finally obtain:

$$(\mathbf{I} - \frac{1}{2}(\mathbf{I} - (\mathbf{I} - \lambda\mathbf{G})\Delta((\mathbf{I} - \lambda\mathbf{G})^{-1})))\mathbf{p} = \frac{1}{2}\mathbf{1}. \tag{20}$$

The equilibrium prices can thus be computed as the solutions to a system of linear equations, as in Equation (1) with

$$\begin{aligned}\mathbf{A} &= \frac{1}{2}(\mathbf{I} - (\mathbf{I} - \lambda\mathbf{G})\Delta((\mathbf{I} - \lambda\mathbf{G})^{-1}),\\ \mathbf{a} &= \frac{1}{2}\mathbf{1}.\end{aligned}$$

In the next step of the computation, we will write down the coefficients of the matrix $\mathbf{A}$ as polynomials in λ. (Notice that the vector $\mathbf{a}$ does not depend on λ, and can thus be decomposed simply as $\mathbf{a} = \mathbf{a}_0 = \frac{1}{2}\mathbf{1}$.) Recall that the diagonal elements of $\Delta((\mathbf{I} - \lambda\mathbf{G})^{-1})$ are $c_{ii} = 1 + \sum_{k=1}^{\infty} \lambda^k g_{ii}^k$. Hence,

$$\begin{aligned}2\mathbf{A} &= \mathbf{I} - (\mathbf{I} - \lambda\mathbf{G})(\mathbf{I} + \sum_{k=1}^{\infty} \lambda^k \Delta(\mathbf{G}^k)), &(21)\\ &= \mathbf{I} - \mathbf{I} - \sum_{k=1}^{\infty} \lambda^k \Delta(\mathbf{G}^k) + \lambda\mathbf{G} + \sum_{k=1} \lambda^{k+1}\mathbf{G}\Delta(\mathbf{G}^k), &(22)\\ &= \sum_{k=1}^{\infty} \lambda^k (\mathbf{G}(\Delta(\mathbf{G}^{k-1})) - \Delta(\mathbf{G}^k)). &(23)\end{aligned}$$

We can thus decompose the matrix of local effects $\mathbf{A}$ using the formal power series:

$$\mathbf{A} = \sum_{l=1}^{\infty} \lambda^l \mathbf{A}_l \qquad (24)$$

where $\mathbf{A}_l = \frac{1}{2}(\mathbf{G}(\Delta(\mathbf{G}^{l-1})) - \Delta(\mathbf{G}^l))$, and construct accordingly the vectors

$$\mathbf{c}_k = \sum_{(p_r)\in\mathcal{P}(k)} \prod_r \mathbf{A}_{p_r}\mathbf{1}. \qquad (25)$$

The following table computes the first two elements of the sequences $\mathbf{c}$ and $\mathbf{c}_i$.

k	$\mathbf{c}^k$	$\mathbf{c}_i^k$
0	$\frac{1}{2}\mathbf{1}$	$\frac{1}{2}$,
1	$\frac{1}{2}\mathbf{d}$	$\frac{1}{2}\deg i$,
2	$\frac{1}{2}(\mathbf{G}^2\mathbf{1} - \mathbf{d})$	$\frac{1}{2}\sum_j g_{ij}(\deg j - 1)$,

This table allows us to show:

Proposition 3.4 *Suppose that firms compete in prices in a network with positive externalities. There exists* $\overline{\lambda} > 0$ *such that, for all* $\lambda < \overline{\lambda}$*, the pricing game admits a unique interior equilibrium* $\mathbf{p}$*. For any two nodes* i,j*,* $p_i > p_j$ *if* $\deg i > \deg j$*. If* $\deg i = \deg j$ *then* $p_i > p_j$ *if* $\sum_k g_{ik} \deg k > \sum_k g_{jk} \deg k$*.*

Proposition 3.4 allows us to compare the prices charged at different nodes. The computation of the sequence $\mathbf{c}_k$ shows that, at the first order, the relevant characteristic of node i is its degree. Prices will be higher for consumers at nodes with larger degrees. This result can easily be interpreted. With positive externalities, demand is higher at nodes with higher degree, so that the best-response of a firm serving a higher degree node is higher. In equilibrium, this translates into a higher price charged at nodes with higher degree. If two nodes have the same degree, the next component to consider in the lexicographic ordering is the sum of the degree of the agent's neighbors: the higher this measure is, the higher the price charged to consumers.

3.2.3 Sensitivity analysis

Proposition 3.4 compares prices at different nodes for low values of the externalities parameter λ. In order to appreciate the limitations of this computation, we ran simulations to investigate the range of externality parameters for which the exact ranking of prices coincides with the ranking in Proposition 3.4.[9] The following Table lists our results. For different numbers of agents ($n = 6, 7, 8, 9, 10, 15$ and 20), we generated 1000 random networks, and computed for each network the threshold value $\overline{\lambda}$ such that the ranking of prices in the network coincides with the ranking obtained by our asymptotic calculations. The table lists the maximal, minimal and mean values of $\overline{\lambda}$ over the 1000 simulations.

n	6	7	8	9	10	15	20
$\overline{\lambda}_{min}$	0.19	0.14	0.01	0.01	0.005	0.01	0.01
$\overline{\lambda}_{max}$	1	1	0.38	0.38	0.305	0.15	0.11
$\overline{\lambda}_{mean}$	0.301	0.248	0.213	0.188	0.160	0.108	0.082

Table 1: Simulations for price rankings

As expected, the threshold value of λ decreases with the number of agents, but remains (in our opinion) surprisingly high, allowing us to claim that the approximation results reflect a robust structural property of the model.

[9] We are immensely grateful to Sebastian Bervoets who wrote the computer program and ran the simulations.

3.2.4 Effects of changes in the network

We now conduct the second exercise of comparative statics to study how changes in the social network g affect equilibrium prices. Given the complexity of the relation between the underlying social network and equilibrium prices captured by Proposition 3.3, we again resort to the asymptotic approach and apply Lemma 2.4.

We consider the effect of the addition of a link ij to a network g. We can use the computations of the first terms of the sequence $\mathbf{c}_k$ to sign the effect of the addition of a link ij on nodes i and j and their neighbors. As both $\deg i$ and $\deg j$ increase, we obtain:

Proposition 3.5 *Suppose that firms compete in prices in a network with positive externalities. There exists $\overline{\lambda} > 0$ such that, for all $\lambda < \overline{\lambda}$, if a new link ij is added to the social network, the prices charged at nodes i, j and any node k such that $g_{ik} = 1$ or $g_{jk} = 1$ strictly increase.*

3.2.5 Multiproduct monopolist

We now suppose that all markets are served by a single firm which chooses discriminatory prices at every node. The objective function of the monopoly is:

$$\Pi^m = \sum_i p_i \mathbf{b}_i(g, \lambda) - c_{ii} p_i^2 - \sum_{j \neq i} c_{ij} p_i p_j. \tag{26}$$

Suppose that λ is small enough so that $c_{ij} < c_{ii}$ for all i, j.[10] Then the matrix $-\mathbf{C}$ is negative semi-definite, and the monopoly's profit is a concave function of the prices $\mathbf{p}$. Hence the optimum is characterized by the first order conditions:

$$c_{ii} p_i + \sum_{j \neq i} c_{ij} p_j = \frac{1}{2} \mathbf{b}_i(g, \lambda), \tag{27}$$

or in matrix terms,

$$(\mathbf{I} - \lambda \mathbf{G})^{-1} \mathbf{p} = \frac{1}{2} (\mathbf{I} - \lambda \mathbf{G})^{-1} \mathbf{1}. \tag{28}$$

Premultiplying both terms of the equation by $(\mathbf{I} - \lambda \mathbf{G})$, we obtain $\mathbf{p} = \frac{1}{2}\mathbf{1}$, and we conclude that *a multiproduct monopolist charges the same price $p = \frac{1}{2}$*

[10] Recall that c_{ij} converges to zero when λ goes to zero, but not c_{ii}.

at each and every node. The price structure of a multiproduct monopolist is thus independent of the network structure.

On the other hand, quantities supplied will vary from node to node, with nodes with higher degree consuming larger quantities. In fact, plugging prices $\mathbf{p} = \frac{1}{2}\mathbf{1}$ in the demand system given in Proposition 3.2, we have

$$\mathbf{q} = \frac{1}{2}\mathbf{b}(g, \lambda). \tag{29}$$

We summarize these findings in the following Proposition.

Proposition 3.6 *A multiproduct monopolist serving all nodes in a network with positive externalities charges the same price at every node. The consumption at node i is proportional to the Bonacich centrality measure of node i with scalar λ.*

Because the monopolist internalizes the consumption externalities across nodes, it should reduce prices at nodes with higher degree and increase prices at nodes with smaller degree, with respect to a firm only serving a single node. The fact that optimal prices are uniform across nodes is an artefact of our linear specification. It can easily be understood by noticing that a quantity setting monopolist should always choose optimal quantities equal to the Bonacich centrality measures of the nodes, implying that prices are uniform across nodes.

3.3 Negative Consumption Externalities

We now consider the case where agents incur negative consumption externalities, so that demand at different nodes are substitutes rather than complements. Let $\mu = -\frac{\lambda}{1+\lambda}$ denote the magnitude of external effects. From the derivation of consumer demands, we can write

$$q_i = \frac{\mathbf{b}_i(\overline{g}, \mu) - \sum_j c_{ij} p_j}{1 + \lambda - \lambda b(\overline{g}, \mu)} \tag{30}$$

where c_{ij} denotes the ij entry of the matrix $[\mathbf{I} - \mu\overline{\mathbf{G}}]^{-1}$.

3.3.1 Price competition

We now conduct the analysis of price competition in a network with negative consumption externalities following the same steps as in the case of positive externalities. The profit of firm i is

$$\Pi_i = \frac{\mathbf{b}_i(\overline{g}, \mu) - c_{ii}p_i^2 - \sum_{j\neq i} c_{ij}p_ip_j}{1 + \lambda - \lambda b(\overline{g}, \mu)}. \tag{31}$$

Instead of expressing equilibrium prices as Bonacich centrality measures of a complicated matrix, we use the asymptotic approach to analyze the relation between prices and nodal characteristics. For μ small enough, the game has a unique interior equilibrium, given by the solution to:

$$2c_{ii}p_i + \sum_{j\neq i} c_{ij}p_j = \mathbf{b}_i(\overline{g}, \mu). \tag{32}$$

or in matrix terms:

$$((\mathbf{I} - \mu\overline{\mathbf{G}})^{-1} + \Delta((\mathbf{I} - \mu\overline{\mathbf{G}})^{-1}))\mathbf{p} = (\mathbf{I} - \mu\overline{\mathbf{G}})^{-1}\mathbf{1}. \tag{33}$$

This expression is exactly equivalent to the expression for positive externalities, with μ replacing λ and $\overline{\mathbf{G}}$ replacing $\mathbf{G}$. Hence, we can compute the first terms of the sequences $\mathbf{c}$ and $\mathbf{c}_i$ as

k	$\mathbf{c}^k$	$\mathbf{c}_i^k$
0	$\frac{1}{2}\mathbf{1}$	$\frac{1}{2}$,
1	$\frac{1}{2}\overline{\mathbf{G}}\mathbf{1}$	$\frac{1}{2}(n - \deg i)$,
2	$\frac{1}{2}(\overline{\mathbf{G}}^2\mathbf{1} - \Delta(\overline{\mathbf{G}}^2)\mathbf{1})$	$\frac{1}{2}\sum_j (1 - g_{ij})(n - \deg j) + n - \deg i$,

This table allows us to show:

Proposition 3.7 *Suppose that firms compete in prices in a network with negative externalities. There exists $\overline{\lambda} < 0$ such that, for all $0 > \lambda \geq \overline{\lambda}$, the pricing game admits a unique interior equilibrium* $\mathbf{p}$. *For any two nodes i,j, $p_i > p_j$ if $\deg i < \deg j$. If $\deg i = \deg j$ then $p_i > p_j$ if $\sum_k (1 - g_{ik})(n - \deg k) > \sum_k (1 - g_{jk})(n - \deg k)$. Furthermore, if a new link ij is added to the social network, the prices charged at nodes i, j and any node k such that $g_{ik} = 1$ or $g_{jk} = 1$ strictly decrease.*

The results of Proposition 3.7 mirror the results of Propositions 3.4 and 3.5 for the case of positive externalities. When externalities are negative, prices charged at nodes with higher degree are lower, and the addition of a new link between i and j reduces the prices at nodes i and j and at their neighbors.

3.3.2 Multiproduct monopolist

A multiproduct monopolist selects a vector of prices $\mathbf{p}$ in order to maximize:

$$\Pi^M = \frac{\sum_i \mathbf{b}_i(\overline{g}, \mu) - c_{ii}p_i^2 - \sum_{j \neq i} c_{ij}p_ip_j}{1 + \lambda - \lambda b(\overline{g}, \mu)}. \tag{34}$$

By an exact parallel to the analysis for positive externalities, we find that the multiproduct monopolist chooses uniform prices across all nodes, and that quantities are lower at nodes with higher degree:

Proposition 3.8 *A multiproduct monopolist serving all nodes in a network with negative externalities charges the same price at every node. The consumption at node i is proportional to the Bonacich centrality measure of node i in the complementary network $\overline{g}$ with scalar $\mu = -\frac{\lambda}{1+\lambda}$.*

4 Price Externalities

We now consider a model where externalities do not result from consumption but from prices. We suppose that agents compare the price they receive with the prices received by their neighbors, and enjoy positive utility if they receive a lower price than the prices in their neighborhood. This psychological effect is likely to play a role when consumers make infrequent purchases of complex goods (like houses, cars or vacation packages) for which they are unable to assess a precise price. By paying less than their neighbors for comparable goods, consumers will perceive that they enjoyed a "good deal" and derive positive utility. This model could either be expressed by assuming that consumers care about the sum of prices charged to their neighbors, or about the average price charged in their neighborhood.

4.1 Average Price Externalities

4.1.1 Consumer demand

We first assume that utilities are defined over the average price charged to a consumer's neighbor:

$$U_i = \theta_i - p_i + \lambda \frac{1}{\deg i} \sum_j g_{ij} p_j. \tag{35}$$

where θ_i is a taste parameter uniformly distributed on $[0, 1]$. A consumer located at node i buys the good if and only if

$$\theta_i \geq p_i - \lambda \frac{1}{\deg i} \sum_j g_{ij} p_j, \tag{36}$$

Hence the demand at node i is given by:

$$q_i = \begin{cases} 0 & \text{if } 1 - p_i + \frac{\lambda}{\deg i} \sum_j g_{ij} p_j < 0, \\ 1 & \text{if } 1 - p_i + \frac{\lambda}{\deg i} \sum_j g_{ij} p_j > 1, \\ 1 - p_i + \frac{\lambda}{\deg i} \sum_j g_{ij} p_j & \text{otherwise} \end{cases}$$

4.1.2 Price competition

If every node is served by a different firm, prices are determined by the Nash equilibrium of a noncooperative game, where firm i sets a price p_i in order to maximize

$$\Pi_i = \begin{cases} 0 & \text{if } 1 - p_i + \frac{\lambda}{\deg i} \sum_j g_{ij} p_j < 0, \\ p_i & \text{if } 1 - p_i + \frac{\lambda}{\deg i} \sum_j g_{ij} p_j > 1, \\ p_i - p_i^2 + \frac{\lambda}{\deg i} \sum_j g_{ij} p_i p_j & \text{otherwise} \end{cases}$$

It is easy to check that the relevant range of prices chosen by firm i is $p_i \in [\lambda \frac{1}{\deg i} \sum_j g_{ij} p_j, \lambda \frac{1}{\deg i} \sum_j g_{ij} p_j + 1]$.[11] If $\lambda = 0$, the game admits a unique interior equilibrium. Hence, by continuity, there exists $\overline{\lambda}$ such that the game admits a unique interior equilibrium for any $\lambda \leq \overline{\lambda}$. In that case, equilibrium prices will satisfy:

$$2p_i - \lambda \frac{1}{\deg i} \sum_j g_{ij} p_j = 1, \tag{37}$$

or in matrix terms:

$$(2\mathbf{I} - \lambda \Delta(\mathbf{id})\mathbf{G})\mathbf{p} = \mathbf{1}. \tag{38}$$

where $\Delta(\mathbf{id})$ is the diagonal matrix with terms $d_i = \frac{1}{\deg i}$. Notice that the matrix $\Delta(\mathbf{id})\mathbf{G}$ is a stochastic matrix. Hence, $\mathbf{1}$ is an eigenvector of the matrix with associated eigenvalue 1, and

[11] These constraints on optimal strategies are different from the positivity constraints of Ballester, Calvó-Armengol and Zenou (2006). As a consequence, the exact bound on λ that they find to guarantee existence of a unique interior equilibrium cannot be directly applied here.

$$(\mathbf{I} - \frac{\lambda}{2}\Delta(\mathbf{id})\mathbf{G})\frac{1}{2-\lambda}\mathbf{1} = \frac{1}{2}\mathbf{1}. \quad (39)$$

We summarize this result in the following Proposition.

Proposition 4.1 *In a network with average price externalities, there exists $\overline{\lambda} > 0$ such that, for all $\lambda < \overline{\lambda}$, competing firms all charge the same price $p = \frac{1}{2-\lambda}$ at every node.*

Proposition 4.1 shows that when consumers experience average price externalities, prices charged by competing firms at different nodes are uniform.[12] Every firm faces a trade-off between raising the price above the price of its competitors (and lowering the demand of its product) or reducing the price below the price of its competitors (and raising the demand of its product). Proposition 4.1 shows that this trade-off is independent of the characteristics of the network, and that every firm, facing the same trade-off, will charge the same price. Hence, consumers at different node will not experience any utility gain or loss from comparing the price they receive with that of their neighbors. However, this does not imply that the model is equivalent to a model without externalities. Due to the presence of externalities, firms will charge higher prices, and it is easy to see that prices are increasing in the externality parameter λ.

4.1.3 Multiproduct monopolist

We now consider the prices chosen by a multiproduct monopolist who internalizes the price externalities experienced by consumers. Noticing that the multiproduct monopolist will always choose prices in the relevant range $p_i \in [\lambda\frac{1}{\deg i}\sum_j g_{ij}p_j, \lambda\frac{1}{\deg i}\sum_j g_{ij}p_j + 1]$, we rewrite her profit as:

$$\Pi^M = \sum_i p_i - p_i^2 + \frac{\lambda}{\deg i}\sum_j g_{ij}p_ip_j. \quad (40)$$

For $\lambda < 1$, the profit function is strictly concave in $\mathbf{p}$, and the optimal prices chosen by the monopolist are uniquely determined by the solution to the system of equations:

$$2p_i - \lambda(\frac{1}{\deg i}\sum_j g_{ij}p_j + \sum_j g_{ij}\frac{1}{\deg j}p_j) = 1, \quad (41)$$

[12]This result also appears in Ghiglino and Goyal (2008)'s analysis when externalities arising from the consumption of status goods depend on the *average* consumption rather than the absolute consumption.

or in matrix terms

$$(\mathbf{I}-\lambda\frac{1}{2}(\Delta(\mathbf{id})\mathbf{G}+\mathbf{G}\Delta(\mathbf{id}))\mathbf{p}=\frac{1}{2}\mathbf{1}. \tag{42}$$

Using the decomposition of Ballester, Calvó-Armengol and Zenou (2006), we can compute the vector of optimal prices as

$$\mathbf{p}=\mathbf{b}(\tilde{g},\frac{\lambda}{2}), \tag{43}$$

where $\tilde{g}$ is the weighted network with adjacency matrix $(\Delta(\mathbf{id})\mathbf{G}+\mathbf{G}\Delta(\mathbf{id}))$. Hence, the multiproduct monopolist charges optimal prices which are equal to the Bonacich centrality measure of the weighted network $\tilde{g}$. However, the relation between nodal characteristics of a node i in the original social network g and the weighted network $\tilde{g}$ are not immediate. In order to shed light on the relation between prices and the characteristics of node i in network g, we again resort to approximation results when λ converges to zero, and apply Lemma 2.3 and 2.4. We compute the first terms of the sequences $(\mathbf{c}^0,...)$ and $(\mathbf{c}_i^0,..)$.

k	$\mathbf{c}^k$	$\mathbf{c}_i^k$
0	$\frac{1}{2}\mathbf{1}$	$\frac{1}{2}$,
1	$\frac{1}{4}(\mathbf{1}+\mathbf{Gid})$	$\frac{1}{4}(1+\sum_j g_{ij}\frac{1}{\deg j})$,
2	$\frac{1}{8}(\mathbf{1}+\mathbf{id}+\Delta(\mathbf{id})\mathbf{G}^2\mathbf{id}+\mathbf{G}\Delta(\mathbf{id})\mathbf{Gid})$	$\frac{1}{8}(1+\frac{1}{\deg i}+\frac{1}{\deg i}(\sum_{j,k} g_{ij}g_{jk}\frac{1}{\deg k})$ $+\sum_j g_{ij}\frac{1}{\deg j}\sum_k g_{jk}\frac{1}{\deg k})$.

Using the computations of the first-order effects, we can easily rank prices at two different nodes and assess the effect of the addition of link ij on the prices p_i and p_j.

Proposition 4.2 *In a network with average price externalities, there exists* $\overline{\lambda}>0$ *such that, for all* $\lambda<\overline{\lambda}$ *the monopolist charges differentiated positive prices. If* $\sum_k g_{ik}\frac{1}{\deg k}>\sum_k g_{jk}\frac{1}{\deg k})$, *the prices verify* $p_i>p_j$. *If* g' *is obtained by adding link* ij *to network* g, *the optimal prices charged by the monopolist satisfy* $p_i'>p_i$ *and* $p_j'>p_j$.

We thus observe that a multiproduct monopolist exploits the average price externality and charges different prices at different nodes. Interestingly, the ranking between p_i and p_j in the first order *does not depend on the degrees of* i *and* j but on the sum of the inverse of the degrees of their neighbors. In words, price p_i will exceed price p_j if consumer i is surrounded by a larger number of neighbors with smaller degrees than consumer j. According to this measure, it is clear that the highest price will be charged to the hub in a star (which has a large number of neighbors with the smallest degree) and the lowest price to a peripheral agent in a star (who has the smallest number of neighbors with the largest degree). Finally, the addition of a new link between i and j does not affect the degree of the other neighbors of i and j but increases the sum of neighbor's inverse degrees and hence has a positive effect on the prices charged at nodes i and j.

4.2 Total price externalities

We now consider the alternative version of the model, where consumers care about *total prices* charged in the neighborhood,

$$U_i = \theta_i - p_i + \lambda \sum_j g_{ij} p_j. \tag{44}$$

By a computation similar to the case of average price externalities, we obtain consumer's demand at node i as:

$$q_i = \begin{cases} 0 & \text{if } 1 - p_i + \lambda \sum_j g_{ij} p_j < 0, \\ 1 & \text{if } 1 - p_i + \lambda \sum_j g_{ij} p_j > 1, \\ 1 - p_i + \lambda \sum_j g_{ij} p_j & \text{otherwise} \end{cases}$$

4.2.1 Price competition

When different firms serve different nodes, in the relevant range of prices, $p_i \in [\lambda \sum_j g_{ij} p_j, \lambda \sum_j g_{ij} p_j + 1]$, profit is given by:

$$\Pi_i = p_i - p_i^2 + \lambda \sum_j g_{ij} p_j. \tag{45}$$

By a direct application of Ballester, Calvó-Armengol and Zenou (2006), we can compute equilibrium prices when λ is small enough.

Proposition 4.3 *In a model with total price externalities, there exists $\overline{\lambda} > 0$ such that, for all $\lambda < \overline{\lambda}$, the noncooperative pricing game admits a unique interior equilibrium and*

$$\mathbf{p} = \mathbf{b}(g, \frac{\lambda}{2}).$$

Proposition 4.3 shows that, as in the model of criminal activities of Ballester, Calvó-Armengol and Zenou (2004) and Calvó-Armengol and Zenou (2004), the decomposition of the matrix of external effects has a transparent interpretation. With total price externalities, prices charged at every node are exactly equal to the Bonacich centrality measure of the node in the social network g with scalar $\frac{\lambda}{2}$. By a direct application of Theorem 2 in Ballester, Calvó-Armengol and Zenou (2006), we also establish that, if a new link ij is added to the social network, the sum of prices charged by the firms increases.

4.2.2 Multiproduct monopolist

If a single firm sells at every node, her profit in the relevant price range is given by:

$$\Pi^M = \sum_i (p_i - p_i^2 + \lambda \sum_j g_{ij} p_j). \tag{46}$$

For small values of λ, the profit function is strictly concave in $\mathbf{p}$, and the optimal prices are characterized by the unique interior solution to the system of equations:

$$1 - 2p_i + 2\lambda \sum_j g_{ij} p_j = 0 \tag{47}$$

We apply Ballester, Calvó-Armengol and Zenou (2006) again to obtain:

Proposition 4.4 *In a model with total price externalities, there exists $\overline{\lambda} > 0$ such that, for all $\lambda < \overline{\lambda}$, the optimal prices chosen by the multiproduct monopolist satisfy*

$$\mathbf{p} = \frac{1}{2}\mathbf{b}(g, \lambda).$$

Proposition 4.4 shows that the ranking of prices across nodes is identical when different firms serve different nodes and when a single firm sells at all nodes. A multiproduct monopolist charges higher prices at nodes with higher Bonacich centrality and benefits from the addition of new links in the network.. Unsurprisingly, the multiproduct monopolist charges higher prices than competing firms at every node.

5 Conclusions

In this paper, we study optimal pricing in networks with quadratic objective functions. We focus on two questions: How do optimal prices reflect the position of agents in the network? What is the effect of a change in the network structure on optimal pricing decisions? Using an asymptotic approach, we show that, when local effects become small, the ranking of optimal prices and strategies can be reduced to the ranking of simple characteristics of the agent's position in the network. In particular, this result shows that with positive consumption externalities, prices are higher at nodes with higher degree, and with relative price externalities, prices are higher at nodes which have more neighbors of smaller degree.

The contribution of this paper is twofold. Our first contribution is methodological: we propose an asymptotic approach to study comparative statics effects which would otherwise be impossible to sign. This asymptotic approach has proved fruitful to analyze prices charged in different settings of consumption with social externalities. It could also be useful to study other linear models where the matrix of interaction is a complex transform of the adjacency matrix of the social network. For example, in Goyal and Moraga-Gonzales (2001)'s model of R & D efforts in networks of strategic alliances, the asymptotic approach shows that firms with higher degree will expand less effort.[13] The second contribution deals with the analysis of oligopoly pricing in social networks. We relate equilibrium and optimal prices to simple characteristics of the nodes, and study when consumers at nodes with higher degree will experience higher or lower prices.

Of course, we are aware of the limitations of our analysis. Our approximations only hold for small local effects, and our method cannot be used to analyze model with large network effects. Simulations are needed to assess the accuracy of our approximation results. In analyzing effects of changes in the network we have focussed attention on connectivity. Following Galeotti and Goyal (2007), we may also look at "second order stochastic dominance" effects, where the number of links in the network is kept fixed, but the variance of the degree distribution is reduced.

6 References

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7 Proofs

Proof of Lemma 2.3: Consider the l_∞ vector norm, defined by:

$$||\mathbf{A}|| = \max_{i,j} |a_{i,j}|. \tag{48}$$

The following Lemma is a direct application of well-known results on matrix norms.

Lemma 7.1 *Suppose that $||\mathbf{A}|| < \frac{1}{n}$. Then, the system of linear equations (1) has a unique solution $\mathbf{x}$ and*

$$||\mathbf{x} - \sum_{k=0}^{K} \mathbf{A}^k \mathbf{a}|| \leq \frac{n^{K+1}||\mathbf{A}||^{K+1}||\mathbf{a}||}{1 - n||\mathbf{A}||}.$$

Proof of Lemma 7.1: Recall that $||\mathbf{A}||$ is *not* a matrix norm, but $n||\mathbf{A}||$ satisfies the submultiplicativity condition, and is indeed a matrix norm (Horn and Johnson (1986), Example 5, p. 322). Hence, if $n||\mathbf{A}|| < 1$, the power series $\sum_k \mathbf{A}^k$ is convergent in one matrix norm, so that $\mathbf{I} - \mathbf{A}$ is invertible and

$$(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k. \tag{49}$$

We thus have:

$$\mathbf{x} = \sum_{k=0}^{K} \mathbf{A}^k \mathbf{a}. \tag{50}$$

Now,

$$
\begin{aligned}
||\mathbf{x} - \sum_{k=0}^{K} \mathbf{A}^k \mathbf{a}|| &= || \sum_{k=K+1}^{\infty} \mathbf{A}^k \mathbf{a}|| && (51)\\
&\leq \sum_{k=K+1}^{\infty} ||\mathbf{A}^k \mathbf{a}|| && (52)\\
&\leq \sum_{k=K+1}^{\infty} n||\mathbf{A}^k||||\mathbf{a}|| && (53)\\
&\leq \sum_{k=K+1}^{\infty} n^k ||\mathbf{A}||^k ||\mathbf{a}||, && (54)\\
&\leq \frac{n^{K+1}||\mathbf{A}||^{K+1}||\mathbf{a}||}{1 - n||\mathbf{A}||}. && (55)
\end{aligned}
$$

The first inequality derives from the triangle inequality of the vector norm, the second from the fact that the matrix norm $n||\mathbf{A}||$ is compatible with the vector norm $||\mathbf{A}||$ (Horn and Johnson (1986), Theorem 5.7.13 p. 324) and the third from the fact that the matrix norm $n||\mathbf{A}||$ is submultiplicative.

Now consider a sequence λ_t of positive scalars converging monotonically to zero. The matrix of cross effects $\mathbf{A}^t$ converges to zero in the l_∞ norm, so there exists $T > 0$ such that $||\mathbf{A}^t|| \leq \frac{1}{n}$ for all $t \geq T$, and, by Lemma 7.1, the system of linear equations possesses a unique interior solution. Furthermore, for $t \geq T$, the series $\sum_{l=1}^{\infty} (\lambda^t)^l \mathbf{A}_l$ and $\sum_{l=0}^{\infty} \lambda_t^l \mathbf{a}_l$ are convergent, so that $\sum_{k=0}^{\infty} \lambda_t^k ||\mathbf{c}^k||$ is a convergent series.

Next, using Lemma 7.1, recall that, for the solution $\mathbf{x}^t$ of the system of equations,

$$
||\mathbf{x}^t - \sum_{k=0}^{K} \lambda_t^k \mathbf{c}^k|| \leq || \sum_{k=K+1}^{\infty} \lambda_t^k \mathbf{c}^k||. \quad (56)
$$

By definition of the l_∞ vector norm, this implies that for all $i = 1, 2, ...n$,

$$
|\mathbf{x}_i^t - \sum_{k=0}^{K} \lambda_t^k \mathbf{c}_i^k| \leq | \sum_{k=K+1}^{\infty} \lambda_t^k \mathbf{c}_i^k|. \quad (57)
$$

Now consider a pair (i, j) and let K be the first element of the sequences $(\mathbf{c}_i^0, ...)$ and $(\mathbf{c}_j^0, ...)$ such that $\mathbf{c}_i^K \neq \mathbf{c}_j^K$. Applying equation (57) to i and j, we obtain,

$$|\frac{x_i^t - x_j^t}{\lambda_t^K} - (\mathbf{c}_i^K - \mathbf{c}_j^K)| \leq 2\lambda_t || \sum_{k=K+1}^{\infty} \lambda_t^{k-K-1} \mathbf{c}_k ||. \tag{58}$$

Now, recall that T is a fixed index of the series, chosen so that the system of equations has a unique interior solution when $t \geq T$. Because the series $\sum_{k=0}^{\infty} \lambda_t^k ||\mathbf{c}^k||$ is convergent, there exists a positive F such that:

$$\sum_{k=0}^{\infty} \lambda_T^k ||\mathbf{c}^k|| \leq F, \tag{59}$$

so that

$$\sum_{k=K+1}^{\infty} \lambda_T^{k-K-1} ||\mathbf{c}^k|| \leq \frac{F}{\lambda_T^{K+1}}. \tag{60}$$

Now,

$$\begin{aligned} || \sum_{k=K+1}^{\infty} \lambda_t^{k-K-1} \mathbf{c}^k || &\leq \sum_{k=K+1}^{\infty} \lambda_t^{k-K-1} ||\mathbf{c}^k||, &(61)\\ &\leq \sum_{k=K+1}^{\infty} \lambda_T^{k-K-1} ||\mathbf{c}^k||, &(62)\\ &\leq \frac{F}{\lambda_T^{K+1}}, &(63) \end{aligned}$$

where the first inequality stems from the properties of the vector norm, the second inequality from the fact that λ_t converges monotonically to zero, and the last inequality from equation (60). Now, this implies that, for any $\epsilon > 0$ there exists $T' > 0$ such that, for all $t \geq \max\{T, T'\}$,

$$2\lambda_t || \sum_{k=K+1}^{\infty} \lambda_t^{k-K-1} \mathbf{c}^k || \leq \epsilon \frac{F}{\lambda_T^{K+1}}, \tag{64}$$

Hence, by inequality (58) the difference between $\frac{x_i^t - x_j^t}{\lambda_t^K}$ and $\mathbf{c}_i^K - \mathbf{c}_j^K$ can be made arbitrarily small, concluding the proof of the Proposition.

Proof of Lemma 2.4: As in the proof of Lemma 2.3, there exists $T > 0$ such that $||\mathbf{A}^t|| \leq \frac{1}{n}$ and $||\mathbf{A}'^t|| \leq \frac{1}{n}$, so that both systems of equations admit unique interior solutions. Let let K be the first element of the sequences $(\mathbf{c}_i^0, ...)$ and $(\mathbf{c}_i'^0, ...)$ such that $\mathbf{c}_i^K \neq \mathbf{c}_i'^K$. Following the same steps as in the proof of Lemma 2.3, one can bound the terms of the series $||\sum_{k=K+1}^{\infty}\lambda_t^{k-K-1}\mathbf{c}^k||$, so that the difference between $\frac{x_i^t - x_i'^t}{\lambda_t^K}$ and $\mathbf{c}_i^K - \mathbf{c}_i'^K$ vanishes, establishing the Proposition.