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Hyper-Stable Social Welfare Functions

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Abstract We introduce a new consistency condition for neutral social welfare functions, called hyper stability. A social welfare function α selects a complete weak order from a profile P of linear orders over any finite set of alternatives. Each linear order p in P generates a linear order over orders of alternatives,called hyper-preference, by means of a preference extension. Hence each profile P generates an hyper-profile $\dot{P}$. We assume that all preference extensions are separable: the hyper-preference of some order p ranks order q above order q' if the set of alternative pairs p and q agree on contains the one p and q' agree on. A special sub-class of separable extensions contains all Kemeny extensions, which build hyper-preferences by using the Kemeny distance criterion. A social welfare function α is hyper stable (resp. Kemeny stable) if at any profile P, at least one linearization of $\alpha(P)$ is ranked first by $\alpha(\dot{P})$, where $\dot{P}$ is any separable (resp. Kemeny) hyper-profile induced from P. We show that no scoring rule is hyper stable, and that no unanimous scoring rule is Kemeny stable, while there exists an hyper stable Condorcet social welfare function.

Key words Hyperpreferences – Kemeny distance – Social Welfare Functions – Stability

1 Introduction

We consider collective choice situations where resolute outcomes are orderings of a finite set of m alternatives, called agendas. Natural examples of agendas are social preference or priority orders over decisions, ranking candidates in sport or arts competition (e.g. the Eurovision song contest), or task assignments. In the latter example, there are m positions to be filled by m candidates, each being assigned a specific position. Given the natural ranking $1 > ... > m$ of candidates, each one-to-one mapping f from the position set to the candidate set induces the order $\succ$ over candidates $f(1) \succ ... \succ f(m)$, although $f(i) \succ f(j)$ does not mean that candidate i is given any kind of priority against candidate j.

Since agendas are resolute outcomes, the classical framework of social choice theory calls for individuals to report their preferences over agendas. Preferences over agendas are orders of orders, and we call them hyper-preferences. However, reporting full preferences faces a problem of practical implementation: in the no-indifference case, individuals have to rank $m!$ outcomes. More generally, when outcomes are complex combinations of basic alternatives, likewise orderings or subsets, choosing from full preference profiles is hardly achievable in practice. This suggests to design procedures based on reduced profiles (or

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simple ballots), which provide only partial information about individual preferences.[1] When outcomes are agendas, a rather natural possibility is asking individuals to report only one agenda, individual agendas being aggregated into one social agenda. Formally, this procedure essentially reduces to using a Social Welfare Function (SWF) α, which maps every profile of linear orders of any number of alternatives into a weak order of alternatives, completed with a tie-breaking rule. Suppose that α is well-defined for any finite number of alternatives. Hence, α provides a weak order from reduced profiles over m alternatives as well as from those over $m!$ alternatives. Furthermore, suppose that α is neutral, meaning that its outcomes are non-sensitive to the names of alternatives. Then, profiles over $m!$ alternatives can be also interpreted as profiles over orders of m alternatives. Choosing from reduced profile clearly entails a huge loss of information about individual preferences. In the spirit of revealed-preference theory, there may nonetheless exist an underlain full preference profile (over $m!$ orders) "compatible" with the reduced profile (over m alternatives), at which α ranks at top the outcome from reduced profile. If this happens at every possible reduced profile, we say that α is hyper-stable. Hyper-stability holds when consistent outcomes prevails at both levels of choice, from full preference profile and from reduced profile, consistency meaning that the outcome from the reduced profile is a best outcome from the full profile.

A key-issue for hyper-stability is what is meant by full profile compatible with reduced profile. We assume here that compatibility holds when hyper-preferences are generated from preferences over alternatives by means of the following criterion: given order P over alternatives, the hyper-preference from P ranks an order Q above another order Q' if the set of alternative pairs P and Q agree on contains the set of pairs P and Q' agree on. All preference extensions satisfying this criterion are called separable. We also pay attention to the sub-domain of separable extensions containing Kemeny preference extensions based on the following Kemeny distance criterion: hyper-preference from P ranks Q above Q' if the Kemeny distance between P and Q is lower than the one between P and Q'.[2] Hyper stability for this sub-domain is called Kemeny stability.

We investigate hyper-stability for the classes of Condorcet SWFs and scoring rules. Our main result is that no scoring rule is Kemeny stable, hence hyper-stable, while there exist unanimous Condorcet SWFs that are hyper-stable. However, many well-known Condorcet SWFs are not hyper stable. Hence, within the class of separable preference extensions, hyper-stability somehow draws a frontier between these two classes of SWFs.

To the best of our knowledge, hyper-stability is a new property for SWFs, although similar notions appear in several studies of collective choice. Using the same setting as above, Laffond and Lainé (2000) characterize the set of (neutral and independent) preference extensions for which whenever pairwise majority comparisons of alternatives in the reduced profile gives an order, this order is a Condorcet winner of the majority tournament among orders built from the full profile. In particular, they show that if reduced profiles involve single-peaked orders, pairwise majority defines an hyper-stable SWF for some preference extension. Özkal-Sanver and Sanver (2006) consider the case of multiple referendum, where outcomes are vectors of yes-or-no positions. If there are more than 2 positions, they show that for the (large) class of separable preferences over outcomes, no issue-wise, unanimous and anonymous voting rule will choose an outcome that would be chosen from preferences over outcomes. This result is generalized by Benoît and Kornhauser (2009) to non-dichotomous issues. Laffond and Lainé (2009) show that in the same framework, choosing issue-wise according to simple majority may lead to an outcome covered in the majority tournament among outcomes.[3] Hence, the Uncovered Set may lead to mutually inconsistent outcomes when applied to different choice levels. In multiple referendum, hyper-preferences are linear orders over the 2^m possible vectors of yes-or-no positions, while reduced profiles involve only one of those vectors.

A work closely related to the present one is due to Binmore (1975), where a stronger notion of hyper-stability is considered. Suppose that agendas are weak orders over three alternatives, and that reduced profiles involve weak orders that are aggregated to a weak order by means of a neutral SWF α. Moreover,

[1] This is what prevails in the Eurovision song contest, where ballots are based on a partial scoring method.

[2] The Kemeny distance between two linear orders is the number of pairs of alternatives which they disagree on.

[3] This result closely relates to the well-known Ostrogorski paradox (Rae and Daudt, 1976, Deb and Kelsey, 1987, Kelly, 1989, Laffond and Lainé, 2006). An outcome x is covered if it is defeated according to majority rule by another outcome which also defeats all outcomes defeated by x. The Uncovered Set is the set of non-covered outcomes.

each weak order generates a partial weak order over weak orders through a lexicographic criterion.[4] In particular, given the 13 possible weak orders over three alternatives, there exist a family $\mathcal{T}$ of triples of agendas on which the criterion generates an hyper-preference. Hence, the reduced profile induces several hyper-preference profiles over triples in $\mathcal{T}$. Since α is neutral, it can be applied to each of these profiles, leading to a weak order R_T over each triple T in $\mathcal{T}$. Furthermore, the weak order chosen from reduced profile also induces a weak order $\widetilde{R}_T$ over each triple T in $\mathcal{T}$. Binmore shows that R_T and $\widetilde{R}_T$ coincide for all T in $\mathcal{T}$ if and only if α is dictatorial, anti-dictatorial or constant. Our definition of hyper-stability is clearly less demanding that Binmore's one, since it only requires that social agendas chosen from reduced profiles are top-ranked from full profiles, imposing nothing about how social agendas generate a social hyper-preference.[5]

Binmore does not comment on hyper-preferences beyond writing "if a rational entity holds a certain preference preordering over a set of alternatives, then that entity must also subscribe to a certain partial preordering of the set of all preorderings" (Binmore, 1975, page 379). We also defend the idea that, provided that orderings are resolute outcomes, individuals must hold preferences over possible outcomes, hence have hyper-preferences. In addition, we advocate that the informational content of real-life ballots is constrained so that hyper-preferences cannot be fully reported, and we question the existence of an hyper-preference profile that essentially brings the same outcome as the one obtained from ballots where individuals report one agenda only.

Hyper-preferences and, at least in watermark, hyper-stability, also appear in the literature of moral judgments. Sen (1974) argues that morality would seem to require a judgment among preferences while rationality would not, and suggests using moral views, defined as hyper-preferences, as a way out of the Paretian liberal paradox.[6]If one accepts reduced profiles as expression of rationality (individuals reporting their first-best outcome[7]) and full profiles as expression of moral judgments, hyper-stability can be interpreted as a property of moral consistency: choice made from rational preferences does not conflict with the one made from moral judgments.

Hyper-stability also closely relates to the property of self-selectivity introduced by Koray (2000) for social choice functions.[8] Loosely speaking, a social choice function is self-selective if it chooses itself against any finite number of other social choice functions. Self-selectivity thus involves two levels of choice: choices from profiles over alternatives, and choices from profiles over choice functions. These two levels are connected by means of a consequentialist principle, which states that individuals preferring alternative x to alternative y will rank any function choosing x above any function choosing y. Koray (2000) shows that a neutral and unanimous social choice function is self-selective if and only if it is dictatorial. We show below that hyper-stability is a necessary condition for self-selectivity of SWF. The argument essentially works as follows: every hyper-preference $\widetilde{Q}$ generated by some order over alternatives Q defines a weak order over SWFs, two SWFs being compared according to the way $\widetilde{Q}$ ranks their respective outcomes. Hence the consequentialist principle applies, but conditional to the choice of a certain preference extension. We say that a SWF is self-selective for some preference extension if, at any profile over alternatives it ranks itself first when compared to any finite set of SWFs. Then, if a SWF α is not hyper stable for some preference extension, there exists a profile P_N over alternatives, together with its induced hyper-preference profile $\dot{P}_N$, at which $\alpha(\dot{P}_N)$ does not rank $\alpha(P_N)$ first. Hence, thanks to consequentialism, α is not self-selective for the Kemeny preference extension.

The rest of the paper is organized as follows. Part 2 formally defines hyper-stability, and investigates its relation to self-selectivity. Hyper-stability of scoring rules is studied in Part 3. In particular, we provide

[4] Weak orders are compared according to their respective top-sets. All top-sets in Binmore's analysis contain at most two elements and the criterion works as follows: GÌven a weak order R, sets $\{x\}$, $\{y\}$, and $\{x, y\}$ are ranked $\{x\} > \{x, y\} > \{y\}$ if and only if xRy.

[5] However, our results can hardly be compared to Binmore's ones, since we do not allow for indifference in individual preferences. Moreover, our setting involves SWFs defined for a variable number of alternatives.

[6] See Igersheim (2007). The reader may refer to Jeffrey (1974), McPherson (1982), and Sen (1977) for further discussion on the general concept of meta-preference.

[7] This approach ignores potential incentives to manipulate the choice procedure. Comments on strategy-proofness are made below.

[8] A social choice function picks one alternative at every profile of preferences over alternatives. For further studies of self-selectivity, see Koray and Unel (2003) and Koray and Slinko (2008).

examples showing that neither the Borda rule, nor the plurality and anti-plurality rules are Kemeny stable, hence hyper stable. Moreover, we show that no unanimous scoring rule is Kemeny stable, and that no scoring rule is hyper stable. Condorcet SWFs are considered in Part 4. We show that the Slater Rule, the Kemeny rule, and the Copeland rule are not hyper-stable. Furthermore, we prove that the transitive closure of the majority relation over alternatives is a unanimous Condorcet hyper-stable SWF. The paper ends up with comments about alternative concepts of hyper-stability, together with open questions. Finally, all proofs are postponed to an appendix.

2 Hyper-stability

2.1 Notations and definitions

Let $\mathbb{N}$ be the set of non-zero natural numbers. We consider societies with variable numbers of individuals and of alternatives. Hence, $\mathbb{N}$ stands for the sets of potential alternatives and individuals, and each actual society involves finitely many individuals confronting finitely many alternatives. Given $m \in \mathbb{N}$, we define $A_m = \{1, ..., m\}$ as a set of m social alternatives. The set of linear (resp. weak) orders over A_m is denoted by $\mathcal{L}(A_m)$ (resp. $\mathcal{R}(A_m)$). An order $P \in \mathcal{L}(A_m)$ is *a linear extension* of $R \in \mathcal{R}(A_m)$ if for any $a, b \in A_m$, $aPb \Rightarrow aRb$. The set of all linear extensions of $R \in \mathcal{R}(A_m)$ is denoted by $\Delta(R)$. Given a set N of n individuals, a weak profile is an element R_N of $\mathcal{R}(A_m)^n$, and a profile is an element P_N of $\mathcal{L}(A_m)^n$. The set of all *linearizations of the weak profile* R_N is $\Delta(R_N) = \times_{i\in N}(\Delta(R_i))$.

A function $F : \cup_{m,n\in\mathbb{N}}\mathcal{L}(A_m)^n \to \cup_{m\in\mathbb{N}}A_m$ is a *social choice function* (SCF) if for all $n, m \in \mathbb{N}$ and all $P_N \in \mathcal{L}(A_m)^n$, $F(P_N) \in A_m$. Furthermore, a function α from $\cup_{m,n\in\mathbb{N}}\mathcal{L}(A_m)^n$ to $\cup_{m\in\mathbb{N}}\mathcal{R}(A_m)$ is a *social welfare function* (SWF) if, for all $n, m \in \mathbb{N}$ and all $P_N \in \mathcal{L}(A_m)^n$, $\alpha(P_N) \in \mathcal{R}(A_m)$. A SCF F is *neutral* if for all $n, m \in \mathbb{N}$ and all $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$, for any permutation γ of A_m, $F(P_N^\gamma) = \gamma(F(P_N))$, where $P_N^\gamma = (P_1^\gamma, ..., P_n^\gamma) \in \mathcal{L}(A_m)^n$ is defined by: $\forall i \in \{1, ..., n\}$, $\forall a, b \in A_m$, aP_ib if and only if $\gamma(a)P_i^\gamma\gamma(b)$. Moreover, a SWF α is neutral if for all $n, m \in \mathbb{N}$ and all $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$, for all $a, b \in A_m$, $a\ \alpha(P_N)\ b$ if and only if $\gamma(a)\ \alpha(P_N^\gamma)\ \gamma(b)$. Note that since F and α are defined for any number of alternatives, neutrality ensures that the precise labelling of alternatives does not matter. In particular, α is defined for profiles over $m!$ alternatives, which can be either basic alternatives or linear orders over m basic alternatives. Furthermore, a SWF α is *unanimous* if, for any $m, n \in \mathbb{N}$, for any profile $P_N \in \mathcal{L}(A_m)^n$, for any two alternatives $a, b \in A_m$, [$a\ P_i\ b$ for all $i = 1, ..., n$] implies that [$a\ \alpha(P_N)\ b$ and $\neg(b\ \alpha(P_N)\ a)$]. Given a SWF α, the *α-induced correspondence* $f_\alpha : \cup_{n,m\in\mathbb{N}}\mathcal{L}(A_m)^n \to 2^A/\varnothing$ is defined by: $\forall n, m \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$, $\forall a \in A_m$, $a \in f_\alpha(P_N) \iff a\ \alpha(P_N)\ b$ for all $b \in A_m$. Hence, the α-induced correspondence selects at each profile P_N the top-set for $\alpha(P_N)$.

2.2 Preference extensions

We turn now to the notion of hyper-preference. A *preference extension* is a function $e : \cup_{m\in\mathbb{N}}\mathcal{L}(A_m) \to \cup_{m\in\mathbb{N}}\mathcal{L}(\mathcal{L}(A_m))$ such that for all $m \in \mathbb{N}$ and all $P \in \mathcal{L}(A_m)$, $e(P) \in \mathcal{L}(\mathcal{L}(A_m))$. Hence, a preference extension maps each linear order over m alternatives to a linear order over all linear orders over alternatives. An element of $\mathcal{L}(\mathcal{L}(A_m))$ is called hyper-preference. An *extension domain* is a proper subset $\mathcal{E}$ of the set of all preference extensions. Given a profile $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$ together with a n-tuple $E = (e_1, ..., e_n) \in \mathcal{E}^n$, an *hyperprofile* of P_N is the element $P_N^E = (e_1(P_1), ..., e_n(P_n))$.

Given $P, Q \in \mathcal{L}(A_m)$, we define the set $A(P, Q) = \{(a, b) \in A_m \times A_m : aPb \text{ and } aQb\}$, which contains all alternative pairs P and Q agree on. We focus on the specific class of *separable* preference extensions.

Definition 1 *A preference extension e is separable if for all $m \in \mathbb{N}$ and all $P, Q, Q' \in \mathcal{L}(A_m)$, $A(P, Q) \supset A(P, Q')$ only if $Q\ e(P)\ Q'$.*

We denote by $\mathcal{S}$ the domain of separable preference extensions.

Given $P, Q \in \mathcal{L}(A_m)$, the *Kemeny distance* between P and Q is defined by $d_K(P, Q) =$ $|\{(a, b) \in A_m \times A_m : aPb \text{ and } bQa\}|$, that is the number of pairs of alternatives P and Q disagree on.

Definition 2 *A preference extension e is Kemeny if for all $m \in \mathbb{N}$ and all $P, Q, Q' \in \mathcal{L}(A_m)$, $d_K(P,Q) < d_K(P,Q')$ only if $Q\ e(P)\ Q'$.*

We denote by $\mathcal{K}$ the domain of Kemeny preference extensions. Pick up any $P \in \mathcal{L}(A_m)$. Using Kemeny distance allows to induce from P the element $\succsim_P \in \mathcal{R}(\mathcal{L}(A_m))$ defined by: $\forall Q, Q' \in \mathcal{L}(A_m)$, $Q \succsim_P Q'$ iff $d_K(P,Q) \leq d_K(P,Q')$, and $Q \succ_P Q'$ iff $d_K(P,Q) < d_K(P,Q')$. In words, the weak order $\succsim_P$ induced by P ranks orders according to their respective distances to P. Given profile $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$, the Kemeny weak profile for P_N is defined by $P_N^K = (\succsim_{P_1}, ..., \succsim_{P_n})$. Thus, a preference extension e is Kemeny if for all $m \in \mathbb{N}$ and all $P \in \mathcal{L}(A_m)$, e is a linear extension of $\succsim_P$. We call Kemeny hyper-profile any linearization of P_N^K. Clearly, every Kemeny extension is separable, and thus $\mathcal{K} \subset \mathcal{S}$.

The Kemeny distance criterion can be criticized by arguing that when comparing two orders, inversions in the lower tail of the ranking are less important that inversions in the upper tail. If three candidates a, b, c are to be ranked as gold, silver and bronze medal, and if your own ranking is $aPbPc$, then you should prefer order $aQcQb$ to order $bQ'aQ'c$, since reversing order for gold and silver seems appears as a more significant deviation than reversing order for silver and bronze. This calls for breaking symmetry by using weighted Kemeny distance (equivalently, this calls for some specific way to break ties in the Kemeny weak profiles). Note however that such a critic no longer holds if agendas are interpreted as task assignments. Indeed, suppose that $aQcQb$ stands for assigning task 1 to individual a task 2 to c and task 3 to b, a similar meaning being given to Q'. Provided that all task are given the same importance, Q and Q' involve only one mismatch from the viewpoint P, and nothing suggests why Q should be preferred to Q'.

The following example illustrates the construction of Kemeny hyper-profiles. Consider the following profile $P_N = (P_1, P_2, P_3)$ over 3 alternatives a, b, c:

$$P_N = \begin{pmatrix} P_1 & P_2 & P_3 \\ \hline a & c & c \\ b & b & a \\ c & a & b \end{pmatrix}$$

The Kemeny weak profile P_N^K of P_N is defined by

$$P_N^K = \begin{pmatrix} \succsim_{P_1} & \succsim_{P_2} & \succsim_{P_3} \\ \hline abc & cba & cab \\ acb, bac & bca, cab & cba, acb \\ bca, cab & acb, bac & abc, bca \\ cba & abc & bac \end{pmatrix}$$

where xyz stands for the linear order $xPyPz$, and where two orders belonging to the same row and column are indifferent. A Kemeny hyper-profile for P_N is any element $\dot{P}_N$ of $\Delta(P_N^K)$. For instance,

$$\dot{P}_N = \begin{pmatrix} \dot{P}_1 & \dot{P}_2 & \dot{P}_3 \\ \hline abc & cba & cab \\ bac & cab & cba \\ acb & bca & acb \\ bca & bac & abc \\ cab & acb & bca \\ cba & abc & bac \end{pmatrix}$$

Contrarily to the Kemeny distance criterion, separability does not automatically induces a weak order over orders. For instance, $e(P_1) \in \mathcal{S}$ only if the following conditions holds: (1) $e(P_1)$ uniquely ranks P_1 first and its inverse cba last, (2) acb is ranked above cab, and (3) bac is ranked above bca. The reader will easily check that hyper-profile $\widetilde{P}_N$ below is built from a vector of separable preference extensions which are not Kemeny.

$$\widetilde{P}_N = \begin{pmatrix} \widetilde{P}_1 & \widetilde{P}_2 & \widetilde{P}_3 \\ \hline abc & cba & cab \\ bac & cab & cba \\ bca & acb & acb \\ acb & bca & abc \\ cab & acb & bca \\ cba & abc & bac \end{pmatrix}$$

2.3 Hyper-stability: definition

We are now ready to formally define hyper-stability:

Definition 3 *Given $n \in \mathbb{N}$, a neutral social welfare function α is hyper-stable for the domain $\mathcal{E}$ of preference extensions if for all $m, n \in \mathbb{N}$, for all $P_N \in \mathcal{L}(A_m)^n$, for all $E = (e_1, ..., e_n) \in \mathcal{E}^n$, we have $\Delta(\alpha(P_N)) \cap f_\alpha(P_N^E) \neq \varnothing$. Moreover, α is Kemeny stable if it is hyper stable for $\mathcal{K}$.*

A neutral SWF α is hyper stable if for domain E if at every finite profile P_N of linear orders over m alternatives, at least one linear extension of the weak order $\alpha(P_N)$ is ranked first by α when applied to any hyper-profile P_N^E induced from P_N by a vector of preference extensions in $\mathcal{E}$.

Figure 1 below illustrates hyper-stability.

Insert Figure 1 here

A society with size n has to rank m alternatives, and has agreed on some SWF α as voting rule. Hence, individual ballots are linear orders of alternatives (profile P_N), and ballots are aggregated by means of α to a weak order $\alpha(P_N)$ of alternatives. Since $\alpha(P_N)$ may involve ties, and since resolute outcomes are linear orders, the final choice results from the use of some tie-breaking rule. The set $\Delta(\alpha(P_N))$ contains all possible outcomes obtained by a tie-breaking rule. Ballots provide little information about preferences over outcomes. We assume that "preferences behind ballots" are induced from ballots by some n-tuple $E = (e_1, ..., e_n)$ of preference extensions. Therefore, the set of ballots P_N together with E generates a profile P_N^E over orders, or hyperprofile. Since α is neutral and defined for any number of alternatives, it can be applied to P_N^E, leading to a weak order $\alpha(P_N^E)$ over outcomes. Hyper-stability prevails for $(e_1, ..., e_n)$ if at least one possible final outcome from ballots is ranked first by α at any full preference profile.

2.4 Hyper-stability and self-selectivity

While the main motivation for studying hyper-stability is that ballots can hardly indicate full preferences over outcomes, another one stems from its close relationship with self-selectivity. Self-selectivity is defined by Koray (2000) for SCFs. Suppose that the society has to choose one alternative among finitely many, as well as the SCF itself. Moreover, suppose that given individual preferences over alternatives, individuals compare SCFs by considering only their respective outcomes. According to this consequentialist principle, initial preferences over alternatives naturally extend to preferences over SCFs: consider any a finite subset $\mathcal{G}$ of neutral SCFs together with a profile $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$; define for all $i = 1, ..., n$ the weak order $R(P_i)$ over $\mathcal{G}$ by: $\forall F, G \in \mathcal{G}$, $F\ R^+(P_i)\ G \Leftrightarrow F(P_N)\ P_i\ G(P_N)$, and $F\ R^\sim(P_i)\ G \Leftrightarrow F(P_N) = G(P_N)$, where $R^+(P_i)$ (resp. $R^\sim(P_i)$) is the a-symmetric (resp. symmetric) part of $R(P_i)$. It follows that P_N induces a dual profile of weak orders $P_N^{\mathcal{G}} = (R(P_1), ..., R(P_n))$ over $\mathcal{G}$. *Self-selectivity* holds for a SCF F if, at any profile over alternatives, F selects itself some linearization of the dual profile over any finite set of SCFs. Formally, F is self-selective if for all $m, n \in \mathbb{N}$, for all $P_N \in \mathcal{L}(A_m)^n$, for all finite subsets $\mathcal{G}$ of neutral SCFs with $F \in \mathcal{G}$, there exists a linearization $\widetilde{P}_N^{\mathcal{G}}$ of $P_N^{\mathcal{G}}$ with $F(\widetilde{P}_N^{\mathcal{G}}) = F$. Koray (2000) proves that, given any fixed size n of the society, a neutral and unanimous SCF is self-selective if and only if its is dictatorial[9].

[9] A SCF F is dictatorial if $\exists 1 \leq i \leq n$ such that, for all $P_N \in \mathcal{L}(A_m)^n$, $F(P_N) = a \Leftrightarrow aP_ib$ for all $b \in A_m/\{a\}$. Moreover, F is unanimous if for any m, for any $P_N \in \mathcal{L}(A_m)^n$, for all $a, b \in A_m$, $[aP_ib$ for all $1 \leq i \leq n] \Rightarrow b \notin F(P_N)$.

Self-selectivity for neutral SWFs is defined along the same lines: at any profile over alternatives, a self-selective SWF ranks itself first among finitely many other SWFs. However, since a SWF provides a weak order, there is no longer a natural duality between preferences over alternatives and preferences over SWFs. In order to make the consequentialist principle meaningful, we need to connect both preference levels by means of a preference extension. It follows that self-selectivity is defined conditional to some domain of preference extensions. This last point is the major difference between the SCF and the SWF settings: choosing preference extensions brings an extra degree of freedom in the analysis, which may allow to escape from Koray's impossibility result.

We formalize self-selectivity for SWFs as follows. A SWF α is called *strict* if for all $n, m \in \mathbb{N}$ and all $P_N \in \mathcal{L}(A_m)^n$, one has $\alpha(P_N) \in \mathcal{L}(A_m)$. A linearization of SWF α is a strict SWF α^* such that for all $n, m \in \mathbb{N}$, for all $a, b \in A_m$ and for all $P_N \in \mathcal{L}(A_m)$, one has , $a\ \alpha^*(P_N)\ b$ only if $a\ \alpha(P_N)\ b$. The set of all linearizations of α is denoted by $L(\alpha)$. Pick up a profile $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$ together with a domain $\mathcal{E}$, and consider any finite subset $\mathcal{A} = \{\alpha_1, ..., \alpha_K\}$ of neutral SWFs. A strict selection of $\mathcal{A}$ is a subset $\mathcal{A}^* = \{\alpha_1^*, ..., \alpha_K^*\}$ of linearizations of $\alpha_1, ..., \alpha_K$. For all $1 \leq i \leq n$, define the weak order $\succsim_{P_i}^{\mathcal{A}^*}$ over $\mathcal{A}^*$ by: $\forall 1 \leq k, k' \leq K$, $\alpha_k^* \succ_{P_i}^{\mathcal{A}^*} \alpha_{k'}^* \Leftrightarrow \alpha_k^*(P_N)\ e_i(P_i)\ \alpha_{k'}^*(P_N)$, and $\alpha_k^* \sim_{P_i}^{\mathcal{A}^*} \alpha_{k'}^* \Leftrightarrow \alpha_k^*(P_N) = \alpha_{k'}^*(P_N)$. Thus, as for SCFs, P_N together with $E = (e_1, ..., e_n) \in \mathcal{E}^n$ induces a dual profile of weak orders $P_N^{E\mathcal{A}^*} = (\succsim_{P_1}^{\mathcal{A}^*}, ..., \succsim_{P_n}^{\mathcal{A}^*})$ over $\mathcal{A}^*$.

Definition 4 *A neutral SWF α is self-selective for the domain of preference extensions E if and only if for all $m, n \in \mathbb{N}$, for all $P_N \in \mathcal{L}(A_m)^n$, for all finite subsets $\mathcal{A}$ of neutral SWFs that contain α, for all strict selection $\mathcal{A}^*$ of $\mathcal{A}$, for any $E = (e_1, ..., e_n) \in \mathcal{E}^n$, there exists a linearization $\widetilde{P}_N^{E\mathcal{A}^*}$ of $P_N^{E\mathcal{A}^*}$ for which $L(\alpha) \cap \mathcal{A}^* \cap\ f_\alpha(\widetilde{P}_N^{E\mathcal{A}^*}) \neq \varnothing$.*

A neutral SWF α is self-selective for domain $\mathcal{E}$ if the following holds: pick up any strict selection $\mathcal{A}^*$ of any finite set $\mathcal{A}$ of neutral SWFs including α, together any profile P_N over alternatives. Picking up n preference extensions in $\mathcal{E}$ generates from P_N a dual profile of weak orders over $\mathcal{A}^*$. Then, there exists a linearization of this dual profile at which α ranks first at least some of its linearizations in $\mathcal{A}^*$.

Note that, although it offers a natural adaptation of the original concept to SWFs, the formalization of self-selectivity sounds complex for two main reasons. First, two different SWFs may have the same outcome at some profile P_N. Therefore, choosing a domain $\mathcal{E}$ is not enough to provide a dual profile of linear orders over SWFs. Second, two SWFs may produce different weak orders at P_N that admit the same linearization. Moreover, note the crucial role played by neutrality, which allows for α to be well-defined for profiles over alternatives and for dual profiles over SWFs.

Proposition 1 below states that hyper-stability is a weaker property than self-selectivity.

Proposition 1 *A neutral SWF is self-selective for domain $\mathcal{E}$ only if it is hyper-stable for $\mathcal{E}$.*

3 Scoring rules

We first study hyper stability of scoring rules. Given a number m of alternatives, a score vector is an element $S^m = (s^{1,m}, s^{2,m}, ..., s^{m,m})$ of $\mathbb{R}_+^m$, where (1) $s^{m,m} = 0$, (2) $s^{1,m} \geq s^{2,m} \geq ... \geq s^{m,m}$, and (3) $s^{1,m} > s^{m,m}$. Given a profile $P_N \in \mathcal{L}(A_m)^n$ together with a score vector S^m, the score of the alternative $x \in A_m$ in P_N is $S^m(x, P_N) = \sum_{i \in N} s^{r_i(x,P_N),m}$, where $r_i(x, P_N)$ is the rank of x in P_i. A SWF α is a *scoring rule* if there exists a sequence $\{S_\alpha^m\}_{m \geq 3} = \{S_\alpha^1, S_\alpha^2, S_\alpha^3 ...\}$ of score vectors such that, for any $m, n \in \mathbb{N}$, for any $P_N \in \mathcal{L}(A_m)^N$, for any two alternatives $x, y \in A_m$, $x\ \alpha(P_N)\ y \iff S_\alpha^m(x, P_N) \geq S_\alpha^m(y, P_N)$. We begin with the analysis of well-known scoring rules, namely the Borda rule, the plurality rule and the anti-plurality rule.

The *Borda rule* $\mathcal{B}$ is defined by: for any $m \in \mathbb{N}$, for any $k \in \{1, ..., m-1\}$, $s_\mathcal{B}^{k,m} = s_\mathcal{B}^{k+1,m} + 1$. It is easily checked that $\mathcal{B}$ is not kemeny stable, hence not hyper stable for $\mathcal{E}$. Indeed, consider the following profile P_N involving 3 alternatives a, b, c and 6 individuals, where the first row indicates the number of individuals sharing the same preference order

$$P_N = \begin{pmatrix} 3 & 1 & 2 \\ \hline a & c & c \\ b & b & a \\ c & a & b \end{pmatrix}$$

Next, consider the following linearization $\dot{P}_N$ of $P_N^{\mathcal{K}}$:

$$\dot{P}_N = \begin{pmatrix} 3 & 1 & 2 \\ \hline abc & cba & cab \\ bac & cab & cba \\ acb & bca & acb \\ bca & bac & abc \\ cab & acb & bca \\ cba & abc & bac \end{pmatrix}$$

Finally, $\mathcal{B}(P_N) = \{acb\} = \Delta(\mathcal{B}(P_N))$, whereas $S^6_{\mathcal{B}}(acb, \dot{P}_N) = 16 < S^6_{\mathcal{B}}(abc, \dot{P}_N) = 19$ implies that $acb \notin f_{\mathcal{B}}(\dot{P}_N)$. Since $\Delta(\mathcal{B}(P_N)) \cap f_{\mathcal{B}}(\dot{P}_N) = \emptyset$, then $\mathcal{B}$ is not Kemeny stable.

The *plurality rule* is the scoring rule π, where, for any $m \in \mathbb{N}$, $s_\pi^{k,m} = 0$ for any $k = 2, ..., m$, and $s_\pi^{1,m} = 1$. Consider the same profile P_N as above. Then $\pi(P_N) = \{acb\}$, while, for any linearization $\dot{P}_N$ of P_N^K, $f_\pi(\dot{P}_N) = \{abc\}$. Hence, π is not Kemeny stable.

The *anti-plurality rule* is the scoring rule λ, where, for any $m \in \mathbb{N}$, $S_\lambda^{k,m} = 1$ for any $1 \leq k \leq m-1$. Consider the following profile $P_N \in \mathcal{L}(A_3)^{15}$, where $\lambda(P_N) = \{abc\}$, together with its associated Kemeny weak profile P_N^K:

$$P_N = \begin{pmatrix} 3 & 2 & 3 & 3 & 4 \\ \hline a & a & b & c & c \\ b & c & a & a & b \\ c & b & c & b & a \end{pmatrix} \quad P_N^K = \begin{pmatrix} 3 & 2 & 3 & 3 & 4 \\ \hline abc & acb & bac & cab & cba \\ acb, bac & abc, cab & abc, bca & cba, acb & cab, bca \\ bca, cab & cba, bac & acb, cba & abc, bca & bac, acb \\ cba & bca & cab & bac & abc \end{pmatrix}$$

We conclude that, for all $P \in \mathcal{L}(A_6)/\{abc\}$, $P\ \lambda(\dot{P}_N)\ abc$ for all $\dot{P}_N \in \Delta(P_N^K)$. Thus, $abc \notin f_\lambda(\dot{P}_N)$, which implies that λ is not Kemeny stable.

We state below four negative results about Kemeny stable scoring rules. The key-ingredient of proofs is the following Theorem, which characterizes Kemeny stable scoring rules for 3 alternatives.

Theorem 1 *A scoring rule α is Kemeny stable if only if $s_\alpha^{1,3} = 2.s_\alpha^{2,3} > 0$, and $s_\alpha^{1,6} = \frac{4}{3}s_\alpha^{2,6} = \frac{4}{3}s_\alpha^{3,6} = 4s_\alpha^{4,6} = 4s_\alpha^{5,6} > s_\alpha^{6,6} = 0$.*

A scoring rule α is *non-truncated* if there exists no $m \in \mathbb{N}$ and no $k \in \{2, ..., m-1\}$ such that $s_\alpha^{k,m} = 0$: the score vector defined for some number m of alternatives gives a strictly positive score to any rank above the last one.

Theorem 2 *There is no Kemeny stable and non-truncated scoring rule.*

A scoring rule α is *strict-at-top* if, for any $m \in \mathbb{N}$, $s_\alpha^{1,m} > s_\alpha^{2,m}$: all score vectors give a score to the top-ranked alternative strictly higher than any other score. Typical examples of strict-at-top scoring rules are the plurality and the Borda rules. Note that any convex scoring rule is also strict-at-top[10].

Theorem 3 *There is no Kemeny stable and strict-at-top scoring rule.*

Since a unanimous scoring rule must be strict-at-top and non-truncated, we can state the following corollary of Theorems 2 and 3.

Theorem 4 *There is no Kemeny stable and unanimous scoring rule.*

When enlarging the Kemeny domain $\mathcal{K}$ to the domain $\mathcal{S}$ of separable preference extensions, we get an even stronger negative result:

Theorem 5 *No scoring rule is hyper stable for $\mathcal{S}$.*

[10] A scoring rule α is convex if, for any $m \in \mathbb{N}$, the score vector $S_\alpha^m = (s_\alpha^{1,m}, ..., s_\alpha^{m,m})$ is such that $(s_\alpha^{1,m} - s_\alpha^{2,m}) \geq (s_\alpha^{2,m} - s_\alpha^{3,m}) \geq ... \geq (s_\alpha^{m-1,m} - s_\alpha^{m,m})$.

4 Condorcet social welfare functions

We turn now to the analysis of Condorcet SWFs. We begin with some additional notations and definitions. Given a profile $P_N \in \mathcal{L}(A_m)^n$, where n is odd, the majority tournament for P_N is the complete and asymmetric binary relation $\mu(P_N)$ defined over $A_m \times A_m$ by: $\forall(x,y) \in A_m \times A_m$, $x\ \mu(P_N)\ y \Leftrightarrow |\{i \in N : xP_iy\}| > |\{i \in N : yP_ix\}|$. A SWF α is *Condorcet* if, for any $m \in \mathbb{N}$, for any $n \in 2\mathbb{N}+1$, for any $P_N \in \mathcal{L}(A_m)^n$, $\mu(P_N) \in \mathcal{L}(A_m) \Rightarrow \alpha(P_N) = \mu(P_N)$. Given $B \subseteq A_m$, the *Condorcet winner* of $\mu(P_N)$ for B is the element $CW(P_N|_B) \in B$ such that $CW(P_N|_B)\ \mu(P_N)\ a$ for all $a \in B/CW(P_N|_B)$, where $P_N|_B$ stands for the restriction of P_N to B. Furthermore, we write $CW(P_N|_{A_m}) = CW(P_N)$. The *transitive closure* $\theta(P_N)$ of $\mu(P_N)$ is defined by: $\forall x,y \in A_m$, $x\theta(P_N)y$ if and only if there exist $x_1, x_2, ..., x_H \in A_m$ such that $x\mu(P_N)x_1$, $x_1\mu(P_N)x_2$, ... , $x_H\mu(P_N)y$.

We prove below the existence of a Condorcet SWF hyper stable for $\mathcal{S}$. Beforehand, we show that three well-known Condorcet SWFs violate Kemeny stability. The *Copeland solution* is the SWF φ defined by: $\forall m \in \mathbb{N}$, $\forall n \in 2\mathbb{N}+1$, $\forall\ P_N \in \mathcal{L}(A_m)^n$, $\forall x,y \in A_m$, $x\ \varphi(P_N)\ y \Leftrightarrow c(x,P_N) \geq c(y,P_N)$, where $c(x,P_N) = |\{z \in A_m : x\ \mu(P_N)\ z\}|$. Consider the following profile P_N, together with the linearization $\dot{P}_N$ of P_N^K:

$$P_N = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \hline a & a & b & b & c \\ b & c & c & a & a \\ c & b & a & c & b \end{pmatrix} \quad \dot{P}_N = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \hline abc & acb & bca & bac & cab \\ acb & cab & bac & bca & acb \\ bac & abc & cba & abc & cba \\ cab & cba & cab & acb & bca \\ bca & bac & abc & cba & abc \\ cba & bca & acb & cab & bac \end{pmatrix}$$

Then, we have $\varphi(P_N) = abc$, while $c(abc, \dot{P}_N) = 3 < c(acb, \dot{P}_N) = 4$ implies that $\Delta(\varphi(P_N)) \cap f_\alpha(\dot{P}_N) = \varnothing$. Thus, φ is not Kemeny stable.

The *Slater solution* is the social welfare correspondence[11] β defined by: $\forall m \in \mathbb{N}$, $\forall n \in 2\mathbb{N}+1$, $\forall\ P_N \in \mathcal{L}(A_m)^n$, $\forall P \in \mathcal{L}(A_m)$, $\beta(P_N) = ArgMin_{P\in\mathcal{L}(A_m)} d_K(P, \mu(P_N))$. A SWF α is *Slater-consistent* if, at any profile P_N, it always selects one linear order in $\beta(P_N)$. Consider the following profile $P_N \in \mathcal{L}(A_8)^5$:

$$P_N = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \hline b & a & d & c & d \\ c & b & a & a & b \\ d & c & b & d & c \\ a & d & c & b & a \\ a' & b' & d' & d' & c' \\ b' & c' & a' & b' & a' \\ c' & d' & b' & c' & d' \\ d' & a' & c' & a' & b' \end{pmatrix}$$

Define $X = \{a,b,c,d\}$ and $Y = \{a', b', c', d'\}$ and consider the restrictions $P_N|_X$ and $P_N|_Y$ of P_N to X and Y respectively. We have that $\mu(P_N|_X)$ and $\mu(P_N|_Y)$ are isomorphic. Moreover, we observe that (1) $a\mu(P_N)b\mu(P_N)c\mu(P_N)d\mu(P_N)a$, (2) $c\mu(P_N)a$, (3) $d\mu(P_N)b$, and (4) all alternatives in X defeat all alternatives in Y . This ensures that $\beta(P_N|_X) = \{cdab\}$ and $\beta(P_N|_Y) = \{c'd'a'b'\}$. Thus, $\beta(P_N) = \{cdabc'd'a'b'\}$. Now, consider $Q = dbcad'b'c'a'$. The next table gives the Kemeny distances between each of the 5 linear orders in $P = (P_1, ..., P_5)$ and respectively, $\beta(P_N)$ and Q:

P_i	$\beta(P_N)$	Q
P_1	$3+4$	$2+5$
P_2	$4+3$	$5+2$
P_3	$3+3$	$2+2$
P_4	$1+3$	$4+0$
P_5	$3+1$	$0+4$

[11] A *social welfare correspondence* is a mapping δ from $\underset{n,m\in\mathbb{N}}{\cup}\mathcal{L}(A_m)^n$ to $\underset{m\in\mathbb{N}}{\cup} 2^{\mathcal{R}(A_m)}$ such that, for any $n, m \in \mathbb{N}$, for any $P_N \in \mathcal{L}(A_m)^n$, $\delta(P_N) \in 2^{\mathcal{R}(A_m)}$, where $2^{\mathcal{R}(A_m)}$ is the set of all non-empty subsets of weak orders over A_m.

It follows that in the Kemeny weak profile P_N^K, Q is strictly preferred to $\beta(P_N)$ by individual 3, while all other individuals are indifferent. Hence, there exists a linearization $\dot{P}_N$ of P_N^K where Q is unanimously preferred to $\beta(P_N)$. Since the Slater solution is contained in the Pareto set, and since $\beta(P_N)$ is a singleton, we conclude that no Slater-consistent SWF is Kemeny stable.

The *Kemeny rule* is the Condorcet social welfare correspondence ω defined by: $\forall\ P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$, $\forall P \in \mathcal{L}(A_m)$, $\omega(P_N) = ArgMin_{P \in \mathcal{L}(A_m)} \sum_{i \in N} d_K(P, P_i)$. A SWF α is *Kemeny-consistent* if, for any profile P_N, it always selects a linear order in $\omega(P_N)$. Consider the following profile $P_N \in \mathcal{L}(A_3)^9$ together with the linearization $\dot{P}_N$ of P_N^K:

$$P_N = \begin{pmatrix} 2 & 3 & 4 \\ \hline b & c & a \\ c & a & b \\ a & b & c \end{pmatrix} \quad \dot{P}_N = \begin{pmatrix} 2 & 3 & 4 \\ \hline bca & cab & abc \\ cba & cba & acb \\ bac & acb & bac \\ cab & bca & cab \\ abc & abc & bca \\ acb & bac & cba \end{pmatrix}$$

The reader will check that $\varphi(P_N) = \{abc\}$, whereas $\varphi(\dot{P}_N) = \{(cab)(abc)(acb)(bca)(cba)(bac)\}$ which leads to $f_\omega(\dot{P}_N) = \{cab\}$. Hence, there is no Kemeny stable and Kemeny-consistent SWF.

We now establish the existence of a Condorcet and unanimous SWF which is hyper stable for $\mathcal{S}$. The *transitive closure* $\theta(P_N)$ of $\mu(P_N)$ is defined by: $\forall x, y \in A_m$, $x\theta(P_N)y$ if and only if there exist $x_1, x_2, ..., x_H \in A_m$ such that $x\mu(P_N)x_1$, $x_1\mu(P_N)x_2$, ... , $x_H\mu(P_N)y$. Consider the SWF θ which maps every profile $P_N \in \cup_{m,n}\mathcal{L}(A_m)^n$ (where n is odd) to the transitive closure $\theta(P_N)$ of $\mu(P_N)$. It is easily checked that θ is unanimous.

Theorem 6 *θ is hyper stable for $\mathcal{S}$*

Note that θ is not the only Condorcet SWF that is hyper stable for $\mathcal{S}$. Define the SWF ω by: $\forall m, n \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$, $\omega(P_N) = \mu(P_N)$ if $\mu(P_N) \in \mathcal{L}(A_m)$, and otherwise, $a\ \omega(P_N)\ b$ and $b\ \omega(P_N)\ a$ for all $a, b \in A_m$. Then ω is hyper stable for $\mathcal{S}$. This is an immediate corollary of the following proposition:

Proposition 2 *Let $P_N \in \mathcal{L}(A_m)^n$ be such that $\mu(P_N) \in \mathcal{L}(A_m)$. For any $E \in \mathcal{S}^n$, either $CW(P_N^E)$ does not exist, or $CW(P_N^E) = \mu(P_N)$.*

5 Discussion

Our main result is that no unanimous scoring rule is Kemeny stable, hence hyper stable for the larger domain $\mathcal{S}$ of separable preference extensions. However, the transitive closure of the majority relation is a unanimous Condorcet SWF that is hyper stable for $\mathcal{S}$.

Hyper stability does not draw a clear border between scoring rules and Condorcet SWFs. Indeed, several Condorcet SWFs based on well-known tournament solutions, as well as the Kemeny SWF, are not Kemeny stable. Characterizing the class of Condorcet SWFs hyper stable for $\mathcal{S}$ is an open question worth being addressed. Another open problem is studying hyper stability for non-unanimous scoring rules.

Further open questions relate to alternative concepts of hyper stability.

5.1 Alternative hyper stability concepts

All Condorcet SWFs violate the following property of hyper Condorcet-stability: A SWF α is *hyper Condorcet-stable* if $\forall n, m \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$, $\forall E \in \mathcal{S}^n$, $\alpha(P_N) \in \mathcal{L}(A_m) \Rightarrow [\alpha(P_N) = CW(P_N^E)]$. To see why, consider the following profile $P_N \in \mathcal{L}(A_m)^5$, together with the Kemeny hyper profile $\dot{P}_N \in \Delta(P_N^K)$:

$$P_N = \begin{pmatrix} 1 & 1 & 1 \\ \hline a & a & b \\ b & c & c \\ c & b & a \end{pmatrix} \quad \dot{P}_N = \begin{pmatrix} 1 & 1 & 1 \\ \hline abc & acb & bca \\ bac & cab & cba \\ acb & abc & bac \\ bca & cba & cab \\ cab & bac & abc \\ cba & bca & acb \end{pmatrix}$$

Since $\mu(P_N) = abc$, then $\alpha(P_N) = abc$ for any Condorcet SWF α. However, $\alpha(P_N)$ is defeated in $\mu(\dot{P}_N)$ by cab. An interesting question is whether some Condorcet SWF α satisfies the following weaker version of hyper Condorcet-stability: $\forall n, m \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$ such that $\alpha(P_N) \in \mathcal{L}(A_m)$, there exists $E \in \mathcal{S}^n$ for which $\alpha(P_N) = CW(P_N^E)$.

Remark that, in the Kemeny hyper profile $\dot{P}_N$ above, all three individual preferences are extended through the same linearization of the Kemeny weak order. This common linearization can be defined as a linear order over the permutations of the set $\{1, 2, 3\}$ of ranks. Indeed, given two orders P and $Q = (a_1 a_2 ... a_m)$ in $\mathcal{L}(A_m)$, define $r_P(Q) = (r_P(a_1), ..., r_P(a_m))$ by $\forall h = 1, ..., m$, $r_P(a_h) = |\{b \in A_m : bPa_h\}| + 1$, that is, the rank given to a_h in P. Moreover, given $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$, we say that the hyper-profile $P_N^E = (e_1(P_1), ..., e_n(P_n))$ is *uniform* if there exists a linear order $\succ$ over the permutations of $\{1, ..., m\}$ such that, for any $i = 1, ..., n$, for any $Q, Q' \in \mathcal{L}(A_m)$, $[Q \ e^i(P_i) \ Q' \Leftrightarrow r_{P_i}(Q) \succ r_{P_i}(Q')]$. In the example above, $\succ$ is defined by: $(123) \succ (213) \succ (132) \succ (231) \succ (312) \succ (321)$. We say that a SWF α is *uniformly hyper stable* for $\mathcal{S}$ if $\forall n, m \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$, $\Delta(\alpha(P_N)) \cap f_\alpha(P_N^E)$ for all uniform hyperprofiles P_N^E with $E \in \mathcal{S}^n$.

As a first step towards a complete study of uniform hyper stability, we remark that the Borda rule $\mathcal{B}$ is not uniformly hyper stable. To see why, consider the following profile P_N, together with the Kemeny hyper profile $\dot{P}_N \in \Delta(P_N^K)$:

$$P_N = \begin{pmatrix} 1 & 1 & 2 \\ \hline a & b & c \\ b & a & b \\ c & c & a \end{pmatrix} \quad \dot{P}_N = \begin{pmatrix} 1 & 1 & 2 \\ \hline abc & bac & cba \\ acb & bca & cab \\ bac & abc & bca \\ cab & cba & acb \\ bca & acb & bac \\ cba & cab & abc \end{pmatrix}$$

We get $\mathcal{B}(P_N) = bca$. Moreover, $\dot{P}_N$ is uniform (to see why, consider $(123) \succ (132) \succ (213) \succ (312) \succ (231) \succ (321)$). Finally, $S^6_{\mathcal{B}}(bca, \dot{P}_N) = 11 < S^6_{\mathcal{B}}(cba, \dot{P}_N) = 12$.

5.2 Strategy-proofness

Hyper stability ignores the issue of strategic behavior. The use of separable preference extensions implies that agendas in reduced profiles are ranked first by individuals according to their underlain preferences. It is already known that choosing agendas from full profiles is not immune to manipulation. Bossert and Storcken (1992) prove impossibility results in the case of Kemeny preference extensions.[12] Incorporating incentives to manipulate in the present framework is a challenging track for further results. In particular, can one characterize the class of hyper stable and strategy-proof SWFs?

6 References

Benoît JP, Kornhauser LA (2010) Only Dictatorship is efficient. *Games and Economic Behavior 70:261-270*

Binmore K (1975) An example in group preference. *Journal of Economic Theory 10:377-385*

[12] See Duddy et al. (2010) and Bossert and Sprumont (2012) for related studies or strategy-proof preference aggregation.

Bossert W, Sprumont Y (2012) Strategy-proof preference aggregation. *Cahier du CIREQ No 12-2012, Université de Montréal*

Bossert W, Storcken T (1992) Strategy-proofness of social welfare functions: the use of the Kemeny distance between preference orderings. *Social Choice and Welfare 9:345-360*

Deb R, Kelsey D (1987) On constructing a generalized Ostrogorski paradox: Necessary and sufficient conditions. *Mathematical Social Sciences 14:161-174*

Duddy C, Perote-Pena J, Piggins A (2010) Manipulating an aggregation rule under ordinally fuzzy preferences. *Social Choice and Welfare 34:411-428*

Igersheim H (2007) Du paradoxe libéral-parétien à un concept de métaclassement des préférences. *Recherches économiques de Louvain 73:173-192*

Jeffrey RC (1974) Preferences among preferences. *The Journal of Philosophy 13:377-391*

Kelly JS (1989) The Ostrogorski paradox. *Social Choice and Welfare 6:71-76*

Koray S (2000) Self-selective social choice functions verify Arrow and Gibbard–Satterthwaite theorems. *Econometrica 68:981–995*

Koray S, Unel B (2003) Characterization of self-selective social choice functions on the tops-only domain. *Social Choice and Welfare 20:495–507*

Koray S, Slinko A (2008) Self-selective social choice functions. *Social Choice and Welfare 31:129–149*

Laffond G, Lainé J (2000) Majority voting on orders. *Theory and Decision, 49:251–289*

Laffond G, Lainé J (2006) Single-switch preferences and the Ostrogorski paradox. *Mathematical Social Sciences 52:49-66*

Laffond G, Lainé J (2009) Condorcet choice and the Ostrogorski paradox. *Social Choice and Welfare 32:317-333*

McPherson MS (1982) Mill's moral theory and the problem of preference change. *Ethics 92:252-273*

Özkal-Sanver I, Sanver MR (2006) Ensuring Pareto optimality by referendum voting. *Social Choice and Welfare 27:211-219*

Rae DW, Daudt H (1976) The Ostrogorski paradox: A peculiarity of compound majority decisions. *European Journal of Political Research 4:391-398*

Sen AK (1970) The impossibility of a Paretian liberal. *Journal of Political Economy 78:152-157*

Sen AK (1974) Choice, orderings and morality. *in S. Korner (ed.) Practical Reason, Blackwell, Oxford*

Sen AK (1977) Rational fools: A critique of behavioral foundations of economic theory. *Philosophy and Public Affairs 6:317-344*

7 Appendix: proofs

7.1 Proof of Proposition 1

Let α be a neutral SCW that is self-selective for some domain $\mathcal{E}$. Pick up any profile $P_N \in \mathcal{L}(A_m)^n$ where $\Delta(\alpha(P_N)) = \{Q_1, ..., Q_H\}$, together with any $E = (e_1, ..., e_n) \in \mathcal{E}^n$. Consider the set of SWFs $\mathcal{A} = \{\alpha_1, \alpha_2, ..., \alpha_H, \beta_1, ..., \beta_{m!-H}\}$ such that:

- $\alpha_1(P_N) = Q_1, ..., \alpha_H(P_N) = Q_H$
- $\forall k \neq k' \in \{1, ..., m! - H\}$, $\alpha_k(P_N) \neq \alpha_{k'}(P_N)$
- $\cup_{1 \leq k \leq m!-H} \beta_k(P_N) = \mathcal{L}(A_m) - \Delta(\alpha(P_N))$

Since all elements of $\mathcal{A}$ are strict SWFs, then $\mathcal{A}$ is a strict selection. Moreover, all elements of $\mathcal{A}$ having different outcomes from P_N, then $P_N^{E\mathcal{A}}$ is a profile of linear orders over $\mathcal{A}$. Furthermore, $[\cup_{1 \leq h \leq H} \alpha_h(P_N)] \cup [\cup_{1 \leq k \leq m!-1} \beta_k(P_N)] = \mathcal{L}(A_m)$ implies that $P_N^{E\mathcal{A}}$ is a profile over all $m!$ linear orders, so that $L(\alpha) \cap \mathcal{A} = L(\alpha)$. It follows from definition that $P_N^{E\mathcal{A}}$ is isomorphic to P_N^E. Since α is self-selective for E, then $L(\alpha) \cap f_\alpha(\widetilde{P}_N^{E\mathcal{A}}) \neq \varnothing$. Since $P_N^{E\mathcal{A}^*}$ is isomorphic to P_N^E, the neutrality of α ensures that $\Delta(\alpha(P_N)) \cap f_\alpha(P_N^E) \neq \varnothing$ and the conclusion follows.

7.2 Proof of Theorem 2

We first prove three useful Propositions, each providing a necessary condition for Kemeny stability:

Proposition 3 *A scoring rule α is Kemeny stable only if $s_\alpha^{2,3} > 0$ and $s_\alpha^{1,6} > s_\alpha^{2,6} = s_\alpha^{3,6} > s_\alpha^{4,6} = s_\alpha^{5,6} > s_\alpha^{6,6} = 0$.*

Proposition 4 *A scoring rule α is Kemeny stable only if $s_\alpha^{1,6} = s_\alpha^{2,6} + s_\alpha^{4,6}$.*

Proposition 5 *A scoring rule α is Kemeny stable only if $s_\alpha^{1,3} = 2s_\alpha^{2,3}$.*

7.2.1 Proof of Proposition 3 The proof is organized in six 6 intermediate lemmas:

Lemma 1 *If α is a Kemeny stable scoring rule, then $s_\alpha^{2,3} > 0$.*

Proof: Suppose that $s_\alpha^{2,3} = 0$, and consider the following profile $P_N \in \mathcal{L}(A_3)^{n_1+n_2+n_3+n_4}$, where $n_1 > n_2 > n_3 + n_4$, together with the following linearization $\dot{P}_N$ of P_N^K:

$$P_N = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 \\ \hline a & b & c & c \\ c & c & a & b \\ b & a & b & a \end{pmatrix} \quad \dot{P}_N = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 \\ \hline acb & bca & cab & cba \\ cab & bac & acb & cab \\ abc & cba & cba & bca \\ bac & cab & bca & acb \\ cba & abc & abc & bac \\ bca & acb & bac & abc \end{pmatrix}$$

It follows that $\Delta(\alpha(P_N)) = \{abc\}$. Kemeny stability requires that $S_\alpha^6(abc, \dot{P}_N) = n_1 s_\alpha^{3,6} + (n_2 + n_3)s_\alpha^{5,6} \geq S_\alpha^6(cab, \dot{P}_N) = (n_1 + n_4)s_\alpha^{2,6} + n_2 s_\alpha^{4,6} + n_3 s_\alpha^{1,6}$, hence that $n_1(s_\alpha^{3,6} - s_\alpha^{2,6}) + n_2(s_\alpha^{5,6} - s_\alpha^{4,6}) + n_3(s_\alpha^{5,6} - s_\alpha^{1,6}) \geq n_4 s_\alpha^{2,6}$, which is clearly impossible □

Lemma 2 *If α is a Kemeny stable scoring rule, then $s_\alpha^{2,6} = s_\alpha^{3,6}$ and $s_\alpha^{4,6} = s_\alpha^{5,6}$.*

Proof: Suppose first that $s_\alpha^{1,3} > 2s_\alpha^{2,3}$, and consider $P_N \in \mathcal{L}(A_3)^4$, and $\dot{P}_N \in \Delta(P_N^K)$:

$$P_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & c & b \\ c & b & a & a \end{pmatrix} \quad \dot{P}_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline abc & acb & bca & cba \\ bac & cab & cba & bca \\ acb & abc & bac & cab \\ bca & cba & cab & bac \\ cab & bac & abc & acb \\ cba & bca & acb & abc \end{pmatrix}$$

Since $S_\alpha^3(a, P_N) = 2s_\alpha^{1,3}$, and $S_\alpha^3(b, P_N) = S_\alpha^3(c, P_N) = s_\alpha^{1,3} + 2s_\alpha^{2,3}$, then $\Delta(\alpha(P_N)) = \{abc, acb\}$. Moreover, we have (1) $S_\alpha^6(abc, \dot{P}_N) = S_\alpha^3(acb, \dot{P}_N) = s_\alpha^{1,6} + s_\alpha^{3,6} + s_\alpha^{5,6}$, and (2) $S_\alpha^6(bca, \dot{P}_N) = s_\alpha^{1,6} + s_\alpha^{2,6} + s_\alpha^{4,6}$. Kemeny stability implies from (1) and (2) that $s_\alpha^{3,6} + s_\alpha^{5,6} \geq s_\alpha^{2,6} + s_\alpha^{4,6}$ (3), which in turn leads to $s_\alpha^{2,6} = s_\alpha^{3,6}$ and $s_\alpha^{4,6} = s_\alpha^{5,6}$.

Suppose now that $s_\alpha^{1,3} < 2s_\alpha^{2,3}$, and consider the same profile P_N above and the hyper-profile $\dot{P}'_N \in \Delta(P_N^K)$ obtained from $\dot{P}_N$ by switching in each order alternatives respectively ranked (1) second and third, and (2) fourth and fifth. We get $\Delta(\alpha(P_N)) = \{bca, cba\}$, and we reach the same conclusion as above by a symmetric argument. Finally, suppose that $s_\alpha^{1,3} = 2s_\alpha^{2,3}$, and consider the profile $P_N \in \mathcal{L}(A_3)^{4Z-1}$ below, where $Z > 1$, together with the Kemeny weak profile P_N^K:

$$P_N = \begin{pmatrix} Z & Z & Z-1 & Z \\ \hline a & b & c & c \\ b & a & b & a \\ c & c & a & b \end{pmatrix} \quad P_N^K = \begin{pmatrix} Z & Z & Z-1 & Z \\ \hline abc & bac & cba & cab \\ acb, bac & abc, bca & bca, cab & cba, acb \\ bca, cab & cba, acb & acb, bac & abc, bca \\ cba & cab & abc & bac \end{pmatrix}$$

Then $\alpha(P_N) = abc$. Moreover, there exists $\dot{P}_N \in \Delta(P_N^K)$ such that $S_\alpha^6(abc, \dot{P}_N) = Z(s_\alpha^{1,6} + s_\alpha^{3,6} + s_\alpha^{5,6})$ and $S_\alpha^6(cab, \dot{P}_N) = Zs_\alpha^{1,6} + (Z-1)s_\alpha^{2,6} + Zs_\alpha^{4,6}$. Kemeny stability requires that $s_\alpha^{3,6} + s_\alpha^{5,6} \geq \frac{Z-1}{Z}s_\alpha^{2,6} + s_\alpha^{4,6}$ for all $Z > 1$. Thus, $s_\alpha^{2,6} + s_\alpha^{4,6} \leq s_\alpha^{3,6} + s_\alpha^{5,6}$, and hence $s_\alpha^{2,6} = s_\alpha^{3,6}$ and $s_\alpha^{4,6} = s_\alpha^{5,6}$ □

We assume in the sequel that α is such that $s_\alpha^{2,6} = s_\alpha^{3,6}$ and $s_\alpha^{4,6} = s_\alpha^{5,6}$ (property $(*)$). Clearly, $(*)$ implies that given any profile P_N over 3 alternatives, given any Kemeny stable SWF α, one has $\alpha(\dot{P}_N) = \alpha(\widetilde{P_N})$ for any two $\forall \dot{P}_N, \widetilde{P_N} \in \Delta(P_N^K)$.

Lemma 3 *If α is a Kemeny stable scoring rule, then* $[s_\alpha^{1,6} = s_\alpha^{2,6}] \Rightarrow [s_\alpha^{4,6} = s_\alpha^{5,6} > 0]$.

Proof: Consider the following $P_N \in \mathcal{L}(A_3)^{3Z+W}$ below, where $Z, W \geq 1$ are chosen such that $W < \frac{s_\alpha^{2,3}}{s_\alpha^{1,3}} Z$:

$$P_N = \begin{pmatrix} Z & Z & Z & W \\ \hline a & b & c & a \\ b & a & b & c \\ c & c & a & b \end{pmatrix}$$

Then $\alpha(P_N) = bac$. Furthermore, using $(*)$ together with Kemeny stability and $s_\alpha^{1,6} = s_\alpha^{2,6}$, one must have $S_\alpha^6(bac, \dot{P}_N) = 2Zs_\alpha^{1,6} + (Z+W)s_\alpha^{5,6} \geq S_\alpha^6(abc, \dot{P}_N) = (2Z+W)s_\alpha^{1,6}$. Thus, $s_\alpha^{1,6} \leq \frac{Z+W}{W} s_\alpha^{5,6}$. Finally, since $s_\alpha^{1,6} > 0$, then $s_\alpha^{5,6} > 0$ □

Lemma 4 *If α is a Kemeny stable scoring rule, then* $[s_\alpha^{1,6} = s_\alpha^{2,6}] \Rightarrow [2s_\alpha^{1,3} = 3s_\alpha^{2,3}]$.

Proof: Define the two profiles $P_N \in \mathcal{L}(A_3)^5$ and $P'_N \in \mathcal{L}(A_3)^{3Z+1}$, where $Z > 1$, as follows:

$$P_N = \begin{pmatrix} 2 & 1 & 1 & 1 \\ \hline a & a & c & b \\ b & c & b & c \\ c & b & a & a \end{pmatrix} \quad P'_N = \begin{pmatrix} 2Z & 1 & Z \\ \hline a & c & c \\ b & a & b \\ c & b & a \end{pmatrix}$$

Suppose first that $2s_\alpha^{1,3} > 3s_\alpha^{2,3}$. It follows from $2s_\alpha^{1,3} > 3s_\alpha^{2,3}$ that $\alpha(P_N) = abc$. Using $(*)$, we have $s_\alpha^{1,6} = s_\alpha^{2,6} = s_\alpha^{3,6} \geq s_\alpha^{4,6} = s_\alpha^{5,6}$. Hence, $\forall \dot{P}_N \in \Delta(P_N^K)$, $S_\alpha^6(abc, \dot{P}_N) = 3s_\alpha^{1,6} + s_\alpha^{5,6}$, and $S_\alpha^6(bac, \dot{P}_N) = 3s_\alpha^{1,6} + 2s_\alpha^{5,6}$. Since Kemeny stability requires $S_\alpha^6(abc, \dot{P}_N) \geq S_\alpha^6(bac, \dot{P}_N)$, then we get $s_\alpha^{5,6} = 0$, in contradiction with Lemma 3.

Similarly, suppose that $2s_\alpha^{1,3} < 3s_\alpha^{2,3}$. From $0 < 2s_\alpha^{1,3} < 3s_\alpha^{2,3}$, we get that $\alpha(P'_N) = bac$ for Z large enough. Moreover, $\forall \dot{P}_N \in \Delta(P_N'^K)$, $S_\alpha^6(bac, \dot{P}_N) = Z(2s_\alpha^{1,6} + s_\alpha^{5,6}) < S_\alpha^6(acb, \dot{P}_N) = Z(2s_\alpha^{1,6} + s_\alpha^{5,6}) + s_\alpha^{1,6}$, in contradiction with Kemeny stability □

Lemma 5 *If α is a Kemeny stable scoring rule, then $s_\alpha^{1,6} > s_\alpha^{2,6}$.*

Proof: Suppose that $s_\alpha^{1,6} = s_\alpha^{2,6}$. From Lemma 3 and 4 together with $(*)$, we have $s_\alpha^{1,6} = s_\alpha^{2,6} = s_\alpha^{3,6}$, $2s_\alpha^{1,3} = 3s_\alpha^{2,3}$, and $s_\alpha^{4,6} = s_\alpha^{5,6} > 0$. Then, consider the following profile $P_N \in \mathcal{L}(A_3)^4$:

$$P_N = \begin{pmatrix} 2 & 1 & 1 & 1 \\ \hline a & b & c & c \\ b & a & b & a \\ c & c & a & b \end{pmatrix}$$

Since $S_\alpha^3(a, P_N) = 2s_\alpha^{1,3} + 2s_\alpha^{2,3}$, $S_\alpha^3(b, P_N) = s_\alpha^{1,3} + 3s_\alpha^{2,3}$, and $S_\alpha^3(c, P_N) = 2s_\alpha^{1,3}$, then, using Lemma 1 and Lemma 4, $\alpha(P_N) = abc$. From Kemeny stability, we have that for any $\dot{P}_N \in \Delta(P_N^K)$, $S_\alpha^6(abc, \dot{P}_N) = 3s_\alpha^{1,6} + s_\alpha^{5,6} \geq S_\alpha^6(acb, \dot{P}_N) = 3s_\alpha^{1,6} + 2s_\alpha^{5,6}$. But this implies that $s_\alpha^{5,6} = 0$, in contradiction with Lemma 3 □

Lemma 6 *If α is a Kemeny stable scoring rule, then $s_\alpha^{3,6} > s_\alpha^{4,6}$.*

Proof: Suppose that $s_\alpha^{3,6} = s_\alpha^{4,6}$. It follows from Lemma 2 together with Lemma 5 that $s_\alpha^{1,6} > s_\alpha^{2,6} = s_\alpha^{3,6} = s_\alpha^{4,6} = s_\alpha^{5,6} \geq s_\alpha^{6,6} = 0$. Using Lemma 1, we get the following possible cases:

Case 1: $s_\alpha^{1,3} = s_\alpha^{2,3} > 0$

Consider the 4 following profiles:

$$P_N = \begin{pmatrix} 3 & 2 & 3 & 3 & 4 \\ \hline a & a & b & c & c \\ b & c & a & a & b \\ c & b & c & b & a \end{pmatrix} \quad P'_N = \begin{pmatrix} 1 & 3 & 1 \\ \hline a & b & c \\ b & a & a \\ c & c & b \end{pmatrix} \quad P''_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & a & b \\ c & b & c & a \end{pmatrix} \quad P'''_N = \begin{pmatrix} 2 & 2 & 1 \\ \hline a & b & c \\ c & a & a \\ b & c & b \end{pmatrix}$$

If $s_\alpha^{2,6} > 0$, then $\alpha(P_N) = abc$. Since $S_\alpha^6(abc, \dot{P}_N)$ $= 3s_\alpha^{1,6} + 8s_\alpha^{2,6} < S_\alpha^6(cba, \dot{P}_N) = 4s_\alpha^{1,6} + 8s_\alpha^{2,6}$ for all $\dot{P}_N \in \Delta(P_N^K)$, then α is not Kemeny stable. If $s_\alpha^{2,6} = 0$, then $\alpha(P'_N) = abc$. Since $f^\alpha(\dot{P}_N) = \{bac\}$ for all $\dot{P}_N \in \Delta(P_N'^K)$, then α is not Kemeny stable.

Case 2: $s_\alpha^{1,3} > s_\alpha^{2,3} > 0$

If $s_\alpha^{2,6} > 0$, then $\alpha(P''_N) = abc$. Since $S_\alpha^6(abc, \dot{P}_N)$ $= s_\alpha^{1,6} + 2s_\alpha^{2,6} < S_\alpha^6(bac, \dot{P}_N) = s_\alpha^{1,6} + 3s_\alpha^{2,6}$ for all $\dot{P}_N \in \Delta(P_N''^K)$, then α is not Kemeny stable. Finally, if $s_\alpha^{2,6} = 0$, then $\alpha(P'''_N) = abc$. Since $S_\alpha^6(abc, \dot{P}_N) = 0 < S_\alpha^6(acb, \dot{P}_N) = 2s_\alpha^{1,6}$ for all $\dot{P}_N \in \Delta(P_N'''^K)$, then α is not Kemeny stable. Thus, Kemeny stability requires that $s_\alpha^{3,6} > s_\alpha^{4,6}$ □

By combining the six lemmas above, we get that any Kemeny stable scoring rule α must satisfy (1) $s_\alpha^{1,6} > s_\alpha^{2,6} = s_\alpha^{3,6} > s_\alpha^{4,6} = s_\alpha^{5,6} \geq 0 = s_\alpha^{6,6}$, and (2) $s_\alpha^{1,3} \geq s_\alpha^{2,3} > 0 = s_\alpha^{3,3}$, hence Proposition 3.

7.2.2 Proof of Proposition 4 Suppose that $s_\alpha^{1,3} > s_\alpha^{2,3}$, and consider profiles $P_N, P'_N \in \mathcal{L}(A_3)^4$ below:

$$P_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & c & b \\ c & b & a & a \end{pmatrix} \quad P'_N = \begin{pmatrix} 1 & 2 & 1 \\ \hline a & a & b \\ b & c & c \\ c & b & a \end{pmatrix}$$

Since $S_\alpha^3(a, P_N) = 2s_\alpha^{1,3}$ and $S_\alpha^3(b, P_N) = S_\alpha^3(c, P_N) = s_\alpha^{1,3} + 2s_\alpha^{2,3}$, then $s_\alpha^{1,3} > s_\alpha^{2,3} \Rightarrow \Delta(\alpha(P_N) = \{abc, acb\}$. Using Proposition 3, we get that for any $\dot{P}_N \in \Delta(P_N^K)$, $S_\alpha^6(abc, \dot{P}_N) = s_\alpha^{1,6} + s_\alpha^{2,6} + s_\alpha^{5,6}$, while $S_\alpha^6(bac, \dot{P}_N) = 2s_\alpha^{2,6} + 2s_\alpha^{5,6}$. Therefore, Kemeny stability requires $s_\alpha^{1,6} \geq s_\alpha^{2,6} + s_\alpha^{5,6}$. Similarly, since $S_\alpha^3(a, P'_N) = 3s_\alpha^{1,3}$, $S_\alpha^3(b, P'_N) = s_\alpha^{1,3} + s_\alpha^{2,3}$ and $S_\alpha^3(c, P'_N) = 3s_\alpha^{2,3}$, then $s_\alpha^{1,3} > s_\alpha^{2,3} \Rightarrow \alpha(P'_N) = abc$. For any $\dot{P}'_N \in \Delta(P_N'^K)$, $S_\alpha^6(abc, \dot{P}'_N) = s_\alpha^{1,6} + 2s_\alpha^{2,6} + s_\alpha^{5,6}$, while $S_\alpha^6(acb, \dot{P}'_N) = 2s_\alpha^{1,6} + s_\alpha^{2,6}$. Thus, Kemeny stability requires $s_\alpha^{1,6} \leq s_\alpha^{2,6} + s_\alpha^{5,6}$. Therefore, if $s_\alpha^{1,3} > 2s_\alpha^{2,3}$, then $s_\alpha^{1,6} = s_\alpha^{2,6} + s_\alpha^{5,6}$.

Suppose that $s_\alpha^{1,3} < s_\alpha^{2,3}$, and consider profiles $\widetilde{P}_N \in \mathcal{L}(A_3)^{5Z+1}$, where $Z > 1$, and $\overline{P}_N \in \mathcal{L}(A_3)^4$ below:

$$\widetilde{P}_N = \begin{pmatrix} 2Z & Z+1 & Z & Z \\ \hline a & c & c & b \\ b & b & a & c \\ c & a & b & a \end{pmatrix} \quad \overline{P}_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & c & b \\ c & b & a & a \end{pmatrix}$$

Since $S_\alpha^3(a, \widetilde{P}_N) = 2Zs_\alpha^{1,3} + Zs_\alpha^{2,3}$, $S_\alpha^3(b, \widetilde{P}_N) = Zs_\alpha^{1,3} + (3Z+1)s_\alpha^{2,3}$ and $S_\alpha^3(c, \widetilde{P}_N) = (2Z+1)s_\alpha^{1,3} + Zs_\alpha^{2,3}$, then if Z is chosen large enough, $s_\alpha^{1,3} > s_\alpha^{2,3} \Rightarrow \alpha(P_N) = bca$. Moreover, using again Proposition 3, Kemeny stability implies that for any $Z > 1$ and any $\dot{P}_N \in \Delta(\widetilde{P}_N^K)$, $S_\alpha^6(bca, \dot{P}_N) \geq S_\alpha^6(abc, \dot{P}_N)$. Thus, $Zs_\alpha^{1,6} + (Z+1)s_\alpha^{2,6} + 3Zs_\alpha^{5,6} \geq 2Z(s_\alpha^{1,6} + s_\alpha^{5,6})$, and therefore $s_\alpha^{1,6} \leq (1 + \frac{1}{Z})s_\alpha^{2,6} + s_\alpha^{5,6}$ for all $Z > 1$, leading to $s_\alpha^{1,6} \leq s_\alpha^{2,6} + s_\alpha^{5,6}$. Similarly, we get $\Delta(\alpha(\overline{P}_N)) = \{bca, cba\}$, while for any $\dot{P}_N \in \Delta(\overline{P}_N^K)$, $S_\alpha^6(bca, \dot{P}_N) = S_\alpha^6(cba, \dot{P}_N) = s_\alpha^{1,6} + s_\alpha^{2,6} + s_\alpha^{5,6}$, while $S_\alpha^6(bac, \dot{P}_N) = 2s_\alpha^{2,6} + 2s_\alpha^{5,6}$. Thus, kemeny stability implies $s_\alpha^{1,6} \geq s_\alpha^{2,6} + s_\alpha^{5,6}$. Therefore, if $s_\alpha^{1,3} < 2s_\alpha^{2,3}$, then $s_\alpha^{1,6} = s_\alpha^{2,6} + s_\alpha^{5,6}$.

Finally, suppose that $s_\alpha^{1,3} = 2s_\alpha^{2,3}$ and consider $Q_N \in \mathcal{L}(A_3)^{4Z+3}$ and $Q'_N \in \mathcal{L}(A_3)^{10Z+1}$, where $Z > 1$:

$$Q_N = \begin{pmatrix} Z & Z+1 & Z+1 & Z+2 \\ \hline a & b & b & c \\ c & c & a & a \\ b & a & c & b \end{pmatrix} \quad Q'_N = \begin{pmatrix} 3Z & 3Z+1 & 2Z & 2Z \\ \hline a & b & b & c \\ c & c & a & a \\ b & a & c & b \end{pmatrix}$$

Since $s_\alpha^{1,3} = 2s_\alpha^{2,3}$, then $\alpha(Q_N) = cba$. From Proposition 3, one has for any $\dot{P}_N \in \Delta(Q_N^K)$ that $S_\alpha^6(cba, \dot{P}_N) = (2Z+3)s_\alpha^{2,6} + (2Z+1)s_\alpha^{5,6}$ and $S_\alpha^6(acb, \dot{P}_N) = Zs_\alpha^{1,6} + (Z+2)s_\alpha^{2,6} + (Z+1)s_\alpha^{5,6}$. Kemeny stability implies $s_\alpha^{1,6} \leq (1 + \frac{1}{z})s_\alpha^{2,6} + s_\alpha^{5,6}$, and thus $s_\alpha^{1,6} \leq s_\alpha^{2,6} + s_\alpha^{5,6}$. Furthermore, we have $\alpha(Q'_N) = bca$, while Kemeny stability implies that for any $\dot{P}_N \in \Delta(Q_N'^K)$ that $S_\alpha^6(bca, \dot{P}_N) \geq S_\alpha^6(bac, \dot{P}_N)$. Hence, $(3Z+1)s_\alpha^{1,6} + 2Zs_\alpha^{2,6} + 2Zs_\alpha^{5,6} \geq 2Zs_\alpha^{1,6} + (3Z+1)s_\alpha^{2,6} + 3Zs_\alpha^{5,6}$, leading to $s_\alpha^{1,6} \geq s_\alpha^{2,6} + \frac{Z}{Z+1}s_\alpha^{5,6}$ for all $Z > 1$, Therefore, if $s_\alpha^{1,3} = 2s_\alpha^{2,3}$, then $s_\alpha^{1,6} = s_\alpha^{2,6} + s_\alpha^{5,6}$, and the proof is complete.

7.2.3 Proof of Proposition 5 Consider the following profiles $P_N \in \mathcal{L}(A_3)^{2Z+2}$ and $P'_N \in \mathcal{L}(A_3)^{56Z+1}$, where $Z > 1$:

$$P_N = \begin{pmatrix} Z+1 & 1 & Z \\ \hline a & a & c \\ b & c & b \\ c & b & a \end{pmatrix} \quad P'_N = \begin{pmatrix} 11Z & 28Z & 17Z & 1 \\ \hline a & a & b & c \\ b & c & c & b \\ c & b & a & a \end{pmatrix}$$

Suppose that $s^{1,3}_\alpha < 2s^{2,3}_\alpha$. Then $\alpha(P_N) = bac$ for Z large enough. Moreover, from Proposition 3, $S^6_\alpha(bac, \dot{P}_N) = (Z+1)(s^{2,6}_\alpha + s^{5,6}_\alpha) < S^6_\alpha(acb, \dot{P}_N) = s^{1,6}_\alpha + (Z+1)s^{2,6}_\alpha + Zs^{5,6}_\alpha$ for all $\dot{P}_N \in \Delta(P^K_N)$, in contradiction with Kemeny stability.

Suppose that $s^{1,3}_\alpha > 2s^{2,3}_\alpha$. Then $\alpha(P'_N) = abc$ for Z large enough. Using again Proposition 3, $S^6_\alpha(abc, \dot{P}_N) = 11Zs^{1,6}_\alpha + 28Zs^{2,6}_\alpha + 17Zs^{5,6}_\alpha$ while $S^6_\alpha(acb, \dot{P}_N) = 28Zs^{1,6}_\alpha + 11Zs^{2,6}_\alpha + s^{5,6}_\alpha$ for all $\dot{P}_N \in \Delta(P'^K_N)$. Since $s^{5,6}_\alpha > 0$ from Proposition 3, we get by using Proposition 4, $S^6_\alpha(abc, \dot{P}_N) = 39Zs^{2,6}_\alpha + 28Zs^{5,6}_\alpha < S^6_\alpha(abc, \dot{P}_N) = 39Zs^{2,6}_\alpha + 28Zs^{5,6}_\alpha + s^{5,6}_\alpha$, in contradiction with Kemeny stability.

7.2.4 End of proof of Theorem 1 **(Necessary Part)** Using Propositions 3,4 and 5, it suffices to prove that if α is Kemeny stable, then $s^{2,6}_\alpha = 3s^{5,6}_\alpha$. Consider the following profiles $P_N \in \mathcal{L}(A_3)^{3Z+1}$ and $P'_N \in \mathcal{L}(A_3)^{3Z-4}$, where $Z > 2$:

$$P_N = \begin{pmatrix} 2Z+1 & Z \\ \hline a & b \\ b & c \\ c & a \end{pmatrix} \quad P'_N = \begin{pmatrix} Z-1 & Z-1 & Z-2 \\ \hline a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

Suppose that $s^{2,6}_\alpha > 3s^{5,6}_\alpha$. Since $s^{1,3}_\alpha = 2s^{2,3}_\alpha$ from Proposition 5, then $\alpha(P_N) = abc$. For any $\dot{P}_N \in \Delta(P^K_N)$, we get from Proposition 3 together with Proposition 4 that $S^6_\alpha(abc, \dot{P}_N) = (2Z+1)s^{1,6}_\alpha + Zs^{5,6}_\alpha = (2Z+1)s^{2,6}_\alpha + (3Z+1)s^{5,6}_\alpha$, while $S^6_\alpha(bac, \dot{P}_N) = (3Z+1)s^{2,6}_\alpha$. But since $s^{2,6}_\alpha > 3s^{5,6}_\alpha$, we get $S^6_\alpha(bac, \dot{P}_N) > S^6_\alpha(abc, \dot{P}_N)$ for all $Z > 2$, in contradiction with Kemeny stability.

Suppose that $s^{2,6}_\alpha < 3s^{5,6}_\alpha$. Using again $s^{1,3}_\alpha = 2s^{2,3}_\alpha$ from Proposition 5, we get $\alpha(P'_N) = abc$. For any $\dot{P}_N \in \Delta(P'^K_N)$, we get from Proposition 3 together with Proposition 4 that $S^6_\alpha(abc, \dot{P}_N) = (2Z-2)s^{2,6}_\alpha$, while $S^6_\alpha(cba, \dot{P}_N) = (Z-2)s^{1,6}_\alpha + (2Z-2)s^{5,6}_\alpha = (Z-2)s^{2,6}_\alpha + (3Z-4)s^{5,6}_\alpha$. Thus, $S^6_\alpha(bac, \dot{P}_N) > S^6_\alpha(abc, \dot{P}_N)$ for Z large enough, in contradiction with Kemeny stability. Hence one must have $s^{2,6}_\alpha = 3s^{5,6}_\alpha$, which proves the Necessary Part.

(Sufficiency Part). Consider any $n \in \mathbb{N}$ together with any profile $P_N \in \mathcal{L}(A_3)^n$ having the form

$$P_N = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \\ \hline a & a & b & b & c & c \\ b & c & a & c & a & b \\ c & b & c & a & b & a \end{pmatrix}$$

with $\sum_{h=1}^{6} n_h = n$. Pick up any scoring rule α fulfilling the conditions $(*)$ $s^{1,3}_\alpha = 2s^{2,3}_\alpha > 0$, and $(**)$ $s^{1,6}_\alpha = \frac{4}{3}s^{2,6}_\alpha = \frac{4}{3}s^{3,6}_\alpha = 4s^{4,6}_\alpha = 4s^{5,6}_\alpha > s^{6,6}_\alpha = 0$. We get that:

- $S^3_\alpha(a, P_N) = (2n_1 + 2n_2 + n_3 + n_5)s^{2,3}_\alpha$
- $S^3_\alpha(b, P_N) = (2n_3 + 2n_4 + n_1 + n_6)s^{2,3}_\alpha$
- $S^3_\alpha(c, P_N) = (2n_5 + 2n_6 + n_2 + n_4)s^{2,3}_\alpha$

Moreover, suppose without loss of generality that $s^{5,6}_\alpha = 1$ and $abc \in \Delta(\alpha(P_N))$. It follows that:

- $n_1 + 2n_2 + n_5 \geq n_3 + 2n_4 + n_6$ *(1)*
- $2n_1 + n_2 + n_3 \geq n_4 + n_5 + 2n_6$ *(2)*
- $n_1 + 2n_3 + n_4 \geq n_2 + 2n_5 + n_6$ *(3)*

Now, pick up any $\dot{P}_N \in \Delta(P^K_N)$. Then we get from $(**)$ that:

- $S^6_\alpha(abc, \dot{P}_N) = 4n_1 + 3(n_2 + n_3) + (n_4 + n_5)$
- $S^6_\alpha(acb, \dot{P}_N) = 4n_2 + 3(n_1 + n_5) + (n_3 + n_6)$
- $S^6_\alpha(bac, \dot{P}_N) = 4n_3 + 3(n_1 + n_4) + (n_2 + n_6)$
- $S^6_\alpha(bca, \dot{P}_N) = 4n_4 + 3(n_3 + n_6) + (n_1 + n_5)$
- $S^6_\alpha(cab, \dot{P}_N) = 4n_5 + 3(n_2 + n_6) + (n_1 + n_4)$

- $S^6_\alpha(cba, \dot{P}_N) = 4n_6 + 3(n_4 + n_5) + (n_2 + n_3)$

Then one easily checks that *(3)* $\Rightarrow S^6_\alpha(abc, \dot{P}_N) \geq S^6_\alpha(acb, \dot{P}_N)$, *(1)* $\Rightarrow S^6_\alpha(abc, \dot{P}_N) \geq S^6_\alpha(bac, \dot{P}_N)$, *(1)+(2)* $\Rightarrow S^6_\alpha(abc, \dot{P}_N) \geq S^6_\alpha(bca, \dot{P}_N)$, *(2)+(3)* $\Rightarrow S^6_\alpha(abc, \dot{P}_N) \geq S^6_\alpha(cab, \dot{P}_N)$, and *(1)+(2)+(3)* $\Rightarrow S^6_\alpha(abc, \dot{P}_N) \geq S^6_\alpha(cba, \dot{P}_N)$. Hence $\Delta(\alpha(P_N)) \cap f_\alpha(\dot{P}_N)$, and the proof is complete.

7.3 Proof of Theorem 2

Let α be a non-truncated and Kemeny stable scoring rule. Consider profile $P_N \in \mathcal{L}(A_6)^{A+B+C+1}$, where $A > B > C > 1:$

$$P_N = \begin{pmatrix} A & B & C & 1 \\ \hline a & b & c & f \\ c & c & a & e \\ b & a & b & d \\ d & d & d & c \\ e & e & f & b \\ f & f & e & a \end{pmatrix}$$

Using Theorem 1, and normalizing S^6_α by setting $s^{1,6}_\alpha = 1$, we get $S^6_\alpha(a, P_N) = A + \frac{3}{4}(B + C)$, $S^6_\alpha(b, P_N) = \frac{3}{4}(A + C) + B + \frac{1}{4}$, $S^6_\alpha(c, P_N) = \frac{3}{4}(A + B) + C + \frac{1}{4}$, $S^6_\alpha(d, P_N) = \frac{1}{4}(A + B + C) + \frac{3}{4}$, $S^6_\alpha(e, P_N) = \frac{1}{4}(A+B) + \frac{3}{4}$, and $S^6_\alpha(f, P_N) = \frac{1}{4}C + 1$. Obviously, A,B and C can be chosen to ensure that $\alpha(P_N) = abcdef$. Consider the following Kemeny hyperprofile $\dot{P}_N \in \Delta(P^K_N)$

$$\dot{P}_N = \begin{pmatrix} A & B & C & 1 \\ \hline acbdef & bcadef & cabdfe & fedcba \\ cabdef & cbadef & cabdef & ... \\ abcdef & bacdef & cbadfe & ... \\ ... & bcdaef & cadbfe & ... \\ ... & bcaedf & cabfde & ... \\ ... & bcadfe & acbdfe & ... \\ ... & cabdef & ... & ... \\ ... & abcdef & ... & ... \\ ... & ... & ... & ... \end{pmatrix}$$

We get that $S^{6!}_\alpha(cabdef, \dot{P}_N) = (A + C)s^{2,6!}_\alpha + Bs^{7,6!}_\alpha + s^{z,6!}_\alpha$, where $z < 6!$, whereas $S^{6!}_\alpha(abcdef, \dot{P}_N) = As^{3,6!}_\alpha + Bs^{8,6!}_\alpha + Cs^{w,6!}_\alpha$, where $w > 6$. Finally, Kemeny stability implies that $s^{1,6!}_\alpha = ... = s^{8,6!}_\alpha$, and $s^{z,6!}_\alpha = 0$, which contradicts that α is non-truncated.

7.4 Proof of Theorem 3

The proof is similar to the one above. Consider profile $P_N \in \mathcal{L}(A_6)^9$ below:

$$P_N = \begin{pmatrix} 4 & 3 & 1 & 1 \\ \hline a & b & c & c \\ c & c & a & b \\ b & a & b & a \\ d & d & d & d \\ e & e & e & e \\ f & f & f & f \end{pmatrix}$$

We get $S^6_\alpha(a, P_N) = 4s^{1,6}_\alpha + s^{2,6}_\alpha + 4s^{3,6}_\alpha$, $S^6_\alpha(b, P_N) = 3s^{1,6}_\alpha + s^{2,6}_\alpha + 5s^{3,6}_\alpha$, $S^6_\alpha(c, P_N) = 2s^{1,6}_\alpha + 7s^{2,6}_\alpha$, $S^6_\alpha(d, P_N) = 9s^{4,6}_\alpha$, $S^6_\alpha(e, P_N) = 9s^{5,6}_\alpha$, and $S^6_\alpha(f, P_N) = 0$. If α is Kemeny stable, it follows from Theorem 1 that $s^{2,6}_\alpha = s^{3,6}_\alpha$, which implies that $\Delta(\alpha(P_N)) \subseteq \{P \in \mathcal{L}(A_6) : P = (abc \rightarrow Q)$, where $Q \in \mathcal{L}(\{d, e, f\})\}$. Consider the following Kemeny hyperprofile $\dot{P}_N \in \Delta(P^K_N)$

$$\dot{P}_N = \left(\begin{array}{cccc} 4 & 3 & 1 & 1 \\ \hline acbdef & bcadef & cabdef & cbadef \\ cabdef & cbadef & acbdef & cabdef \\ abcdef & bacdef & cbadef & bcadef \\ ... & bcdaef & cadbef & cbdaef \\ ... & bcaedf & cabedf & cbaedf \\ ... & bcadfe & cabdfe & cbadfe \\ ... & cabdef & abcdef & abcdef \\ ... & abcdef & ... & ... \\ ... & ... & ... & ... \end{array} \right)$$

We get that $S^{6!}_\alpha(cabdef, \dot{P}_N) = 4s^{2,6!}_\alpha + 3s^{7,6!}_\alpha + s^{1,6!}_\alpha + s^{2,6!}_\alpha$, whereas $S^{6!}_\alpha(abcdef, \dot{P}_N) = 4s^{3,6!}_\alpha + 3s^{8,6!}_\alpha + s^{7,6!}_\alpha + s^{7,6!}_\alpha$. Using $s^{2,6!}_\alpha \geq s^{3,6!}_\alpha$ and $s^{7,6!}_\alpha \geq s^{8,6!}_\alpha$ together with the strict-at-top property, we have that $S^{6!}_\alpha(cabdef, \dot{P}_N) > S^{6!}_\alpha(abcdef, \dot{P}_N)$. The conclusion follows from the fact that $abcdef$ maximizes $S^{6!}_\alpha(P, \dot{P}_N)$ in $\Delta(\alpha(P_N))$.

7.5 Proof of Theorem 5

Consider profile $P_N \in \mathcal{L}(A_3)^5$ below:

$$P_N = \left(\begin{array}{cccc} 2 & 1 & 1 & 1 \\ \hline a & b & c & c \\ b & a & b & a \\ c & c & a & a \end{array} \right)$$

Pick up a scoring rule α hyper stable for $\mathcal{S}$. Since $\mathcal{K} \subset \mathcal{S}$, then α is Kemeny stable. It follows from Theorem 1 that score vectors must be such that $(*)$ $s^{1,3}_\alpha = 2s^{2,3}_\alpha > 0$, and $(**)$ $s^{1,6}_\alpha = \frac{4}{3}s^{2,6}_\alpha = \frac{4}{3}s^{3,6}_\alpha = 4s^{4,6}_\alpha = 4s^{5,6}_\alpha > s^{6,6}_\alpha = 0$. It follows that $\alpha(P_N) = abc$. It is straightforward to check that the following hyperprofile P^E_N is built from a 5-tuple $E = (e_1, e_1, e_3, e_4, e_5)$ of separable preference extensions:

$$P^E_N = \left(\begin{array}{cccc} 2 & 1 & 1 & 1 \\ \hline abc & bac & cba & cab \\ bac & bca & bca & acb \\ acb & abc & bac & cba \\ bca & acb & cab & bca \\ cab & cba & acb & abc \\ cba & cab & abc & bac \end{array} \right)$$

Note that all extensions in E but e_4 are Kemeny. We get $S^6_\alpha(abc, P^E_N) = 12s^{5,6}_\alpha < S^6_\alpha(bac, P^E_N) = 13s^{5,6}_\alpha$, which contradicts hyper stability for $\mathcal{S}$.

7.6 Proof of Theorem 6

Given any $Q \in \mathcal{L}(A_m)$, we write $Q = (Q_1 \to Q_2 \to ... \to Q_H)$, where, for any $1 \leq h \leq H$, $Q_h \in \mathcal{L}(B_h)$ is a segment of Q, and where $\{B_1, B_2, ..., B_H\}$ is a partition of A_m into non-empty sets. Lemma 7 is a useful intermediate step towards the proof.

Lemma 7 *Let $Q, Q' \in \mathcal{L}(A_m)$ be respectively defined by $Q = (Q_1 \to x \to Q_2 \to y \to Q_3)$ and $Q' = (Q_1 \to y \to Q_2 \to x \to Q_3)$, where $Q_h \in \mathcal{L}(B_h)$, for $1 \leq h \leq 3$. Then, for any $P_N \in \mathcal{L}(A_m)^n$ with n is odd, and any $E = (e_1, ..., e_n) \in \mathcal{S}^n$, $[x\ \mu(P_N)\ y] \Rightarrow [Q\ \mu(P^E_N)\ Q']$,*

Proof: Define $B = \{x, y\} \cup B_2$, where $Q_2 \in \mathcal{L}(B_2)$. Pick up any $P_i \in \mathcal{L}(A_m)$ where xP_iy, and consider the restriction $P_i|_B$ of P_i to B. We can write $P_i|_B = (V_1 \to x \to V_2 \to y \to V_3)$, where V_1, V_2, and V_3 are segments of $P_i|_B$, with $V_h \in \mathcal{L}(B_{2h})$, $1 \leq h \leq 3$, and $\{B_{21}, B_{22}, B_{23}\}$ being a partition of B_2. Then $A(P_i|_B, Q|_B) = \{x, y\} \cup [\{x\} \times (B_{22} \cup B_{23})] \cup [(B_{21} \cup B_{22}) \times \{y\}] \cup A(P_i|_{B_2}, Q_2)$, while $A(P_i|_B, Q'|_B) = A(P_i|_B, Q|_B)/\{x,y\}$. Hence, $A(P_i|_B, Q'|_B) \subset A(P_i|_B, Q|_B)$. Since Q and Q' have the same segment Q_1 at top and the same segment Q_3 at bottom, then $A(P_i, Q') \subset A(P_i, Q)$. From separability of e_i, we get $Q\ e_i(P_i)\ Q'$. Finally, $x\ \mu(P_N)\ y$ implies that $|\{i : xP_iy\}| > \frac{n}{2}$, hence that $|\{i : Q\ e_i(P_i)\ Q'\}| > \frac{n}{2}$ and the conclusion follows □

Given $P_N \in \mathcal{L}(A_m)^n$, the top-cycle for P_N is the subset $T(B, P_N)$ of A_m containing all maximal elements for $\theta(P_N)$. The transitive closure partition of A_m is the ordered set $S(\theta, P_N) = (S_1, S_2, ..., S_J)$ of indifference classes for $\theta(P_N)$, where $\forall j \leq j' \in \{1, ..., J\}$, $\forall (x, x') \in S_j \times S_{j'}$, $x\theta(P_N)x'$ and $\neg(x'\theta(P_N)x$ if $j < j'$. By definition of θ, one has $\Delta(\theta(P_N)) = \{Q \in \mathcal{L}(A_m) : Q = (Q_1 \to Q_2 \to ... \to Q_J)$ where, for each $j = 1, ..., J$, $Q_j \in \mathcal{L}(S_j)\}$. The proof of Theorem 6 is complete if we show that for any $E = (e_1, ..., e_n) \in \mathcal{S}^n$, $\Delta(\theta(P_N)) \cap T(\mathcal{L}(A_m), P_N^E) \neq \varnothing$.

Pick up any $P \in \mathcal{L}(A_m)/\Delta(\theta(P_N))$ and any $E = (e_1, ..., e_n) \in \mathcal{S}^n$. Define $B(P) = \{x \in A_m : x \in S_j$ for some j and $\forall y \in S_{j'}/\{x\}$, $xPy \Rightarrow j' > j\}$, and $B = A_m/B(P)$. Consider order $Q(P) \in \Delta(\theta(P_N))$ such that:

- $Q(P) \mid_{B(P)} = P \mid_{B(P)}$
- $xPy \Rightarrow xQ(P)y$ for all $x, y \in B \cap S_j$ for some $j \in \{1, ..., J\}$

Write $P \mid_B = b_1b_2...b_T$, where $T = |B|$. There exists a permutation σ of $\{1, ..., T\}$ such that $Q(P) \mid_B = b_{\sigma(1)}b_{\sigma(2)}...b_{\sigma(T)}$. Then, there is a finite sequence $\{\omega_h\}_{1 \leq h \leq H}$ of transpositions of A_m, where $H \leq T$, such that ω_1 swaps b_1 and $b_{\sigma(1)}$ in $P \mid_B$, leading to $P^1 \mid_B = b_{\omega_1(1)}b_{\omega_1(2)}...b_{\omega_1(T)}$, ω_2 swaps $b_{\omega_1(2)}$ and $b_{\sigma(2)}$ in $P^1 \mid_B$, leading to $P^2 \mid_B = b_{\omega_2\circ\omega_1(1)}b_{\omega_2\circ\omega_1(2)}...b_{\omega_2\circ\omega_1(T)}$, ..., ω_H swaps $b_{\sigma(T)}$ and $b_{\omega_{T-1}\circ...\circ\omega_1(T)}$ in $P^{T-1} \mid_B$, leading to $P^T \mid_B = Q(P) \mid_B$. Since $b_{\sigma(1)}\mu(P_N)b_{\sigma(2)}\mu(P_N)...\mu(P_N)b_{\sigma(T)}$, then Lemma 7 ensures that for all $1 \leq h \leq H$, either $P^{h+1} \mid_B = P^h \mid_B$ or $(P^{h+1} \mid_B)\mu(P_N^E \mid_B)(P^h \mid_B)$. Hence $(Q(P) \mid_B)\theta(P_N^E \mid_B)(P \mid_B)$, and thus $Q(P)\theta(P_N^E)P$. This proves that for any order P not in $\Delta(\theta(P_N))$, there exists $Q \in \Delta(\theta(P_N))$ such that $Q\theta(P_N^E)P$.

Finally, since $T(\mathcal{L}(A_m), P_N^E \mid_{\Delta(\theta(P_N))}) \neq \varnothing$, there exists $Q \in \Delta(\theta(P_N))$ such that $Q\theta(P_N^E)Q'$ for all $Q' \in \Delta(\theta(P_N))/\{Q\}$. Thus, there exists $Q \in \Delta(\theta(P_N))$ such that $Q\theta(P_N^E)Q'$ for all $Q' \in \mathcal{L}(A_m)/\{Q\}$. Thus $\Delta(\theta(P_N)) \cap T(\mathcal{L}(A_m), P_N^E)) \neq \varnothing$ and the proof is complete.

7.7 Proof of Proposition 2

Let $P_N = (P_1..., P_i, ..., P_n) \in \mathcal{L}(A_m)^n$. Choose any $E = (e_1, ..., e_n) \in \mathcal{S}^n$. Suppose without loss of generality that $\mu(P_N) = a_1a_2...m$. Moreover, suppose that $CW(P_N^E) = Q = b_1b_2...b_m$, with $b_1 \neq a_1 = b_h$ for some $2 \leq h \leq m$. Now define $Q' = a_1b_2...b_{h-1}b_hb_{h+1}...b_m \in \mathcal{L}(A_m)$. It follows from Lemma 7 above that $[1\ \mu(P_N)\ b_1] \Rightarrow [Q'\ \mu(P_N^E)\ Q']$, which contradicts $CW(P_N^E) = Q$. Thus, $b_1 = a_1$. We conclude by iterating the same argument for $b_2, ..., b_m$.

Figure