

# On Existence in Equilibrium Models with Endogenous Default

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## Abstract

In this paper I point out that models of default in the spirit of Dubey, Geanakoplos and Shubik (2005) can generate equilibria that seem incompatible with competitive behavior on the part of lenders. I show, in fact, that existence in these models only holds universally because the concept allows for equilibria that are return-dominated in the sense that the borrowing rate could be lowered on some contracts in such a way as to make both borrowers and lenders strictly better off. In economies that rule out these unsustainable equilibria, outcomes may involve credit rationing in the sense of Stiglitz and Weiss (1981).

**Preliminary and incomplete, comments welcome.**

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# 1 Introduction

Most loan contracts are written between counterparties who understand that default is a possibility. Lenders take this possibility into account when deciding whether to issue loans and what terms to offer borrowers. In the residential mortgage context for instance, observable characteristics such as income and credit scores affect both approval decisions and interest rates. In general equilibrium therefore, the possibility of default should matter for the aggregate quantity of financial intermediation and the terms at which different agents borrow and lend.

Studying the theoretical implications of the default option requires a general equilibrium model where borrowers have the option to renege on their financial promises and where the consequences of exercising this option are fully specified. Among the most natural choices for this purpose is the seminal model of Dubey, Geanakoplos and Shubik (DGS, 2005.) The DGS framework builds on the standard general equilibrium model with incomplete market by allowing for default. Agents take commodity and asset prices as given as in the standard model, but they also take as given delivery rates hence effective returns on assets.

DGS establish that equilibria always exists in this environment, a remarkable result since endogenizing asset payoffs and allowing heterogenous agents to self-select into financial contracts is known to make existence problematic in competitive models. The universality of this existence result stems for the nature of the equilibrium concept DGS employ: agents take delivery rates as given, and delivery choices by agents must be consistent with those expectations. Existence then boils down to a fairly standard fixed point problem and, under standard conditions on preferences and budget sets, the usual arguments apply (see Bisin et al., 2010, for additional illustrations of the benefits of this equilibrium concept.)

In this paper, I show that while the DGS equilibrium concept guarantees existence, it also tends to generate equilibria that do not seem compatible with competitive behavior on the part of agents. Specifically, since by assumption lenders ignore the impact of loan terms on delivery rates, a particular equilibrium may be such that by lowering stated returns on assets

– by paying a higher price for a given asset, that is – asset buyers could raise effective, net-of-default returns. The intuition for this is simple: raising stated returns can cause delivery rates to fall more than proportionately. The question, then, is what could prevent a small set of lenders from proposing a small set of borrowers terms that would make both parties better off?

DGS argue that assuming that lenders take delivery rates as given is justified in contexts where assets are pooled by intermediaries and sold to large anonymous pools of lenders. That argument seems incomplete, at best. Anonymous as these markets may be, if it is possible to change loan terms across the board to increase cash-flows in all states while making all borrowers better off, it seems that someone – the securitizer, for one – should recognize and take advantage of this costless profit opportunity. The originator and the securitizer benefit either directly (via overcollateralization) or through reputational effects when the pools they generate and securitize perform better. This is true even when a market for default insurance (Credit Default Swaps, say) exist since premia obviously depend on expected performance.<sup>1</sup> It should come as little surprise, then, that large mortgage lenders and the investment banks that purchase and pool existing mortgages invest significant resources in using historical data to forecast the performance of mortgages given borrower and loan characteristics at origination.

To put this in more general terms, price taking can be justified in the standard framework on trivial, compelling grounds: in equilibrium, no asset seller (borrower) would accept a lower prices from a buyer (lender), and buyers have no reason to offer a price higher than the equilibrium price. In the DGS model, while the first part of the standard argument remains correct, buyers may now be able to raise delivery rates by paying a higher price for the asset. Nothing prevents agent from taking advantage of this option when it is present. Competition should eliminate this glaring profit opportunity.

After showing that return-dominated equilibria can arise, I define and study a natural

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<sup>1</sup>In residential mortgage markets, insurance is typically provided by Government Sponsored Enterprises. These agencies would benefit directly by imposing underwriting standards that lead to fewer default. These standards include debt-to-income ratios, i.e. the ratio of the initial payment to the borrower's income.

selection procedure that discards these equilibria. In general however, excluding these equilibria compromises existence as I demonstrate via a simple example. This means that DGS' existence argument depends critically on calling equilibria a set of contract terms such that lenders are choosing to forego clear profit opportunities. In fact, in my counter-example, equilibria must involve some credit-rationing in the sense of Stiglitz and Weiss (1981).

In this sense, my findings favor the conventional wisdom that prevailed before DGS' existence result. DGS (2005, p15) describe this conventional wisdom as “the historical tendency to associate default with disequilibrium . . .” They go on to write that “. . . the endogeneity of the asset payoff structure is known to complicate the existence of equilibrium with incomplete markets. But we show that no new existence problems arise from the endogeneity of the asset payoffs due to default.” One way to interpret my results is that the conventional wisdom is in fact exactly right.

A version of the DGS concept has also been used to study environments where asset purchases must be collateralized.<sup>2</sup> As is well known, many loans used to finance the purchase of a durable good are collateralized by that good. When liability is limited to that collateral and borrowers incur no additional default penalty, they default when and only when the value of their debt exceeds the value of that collateral. This provides a theoretical framework in which the popular notion of “strategic default” can be formalized. In this simpler environment, it should be fairly obvious that lowering stated payoffs cannot increase expected delivery rates and in that sense, the issues I have raised in this paper become less problematic.

Nevertheless, environments where at least as a first approximation borrowers incur no default cost beyond a specific collateral loss are likely to be few and far between. Take for instance the canonical example of residential mortgages. Even in the few US states (California, e.g.) where liability is effectively limited to the home, borrowers incur myriad other costs when they default, including the impact on their credit history and ability to borrow and become home-owners again. In fact, while popular accounts would have one believe that ruthless,

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<sup>2</sup>The pioneering paper is Geanakoplos and Zame (2005). See also Araujo et al. (1998) and Araujo et al. (2005).

strategic default is common, the facts do not bear that out. In the United States, even in the midst of the most severe housing downturn on record, most households with negative equity choose to continue meeting their financial obligations.<sup>3</sup>

This suggests that residential borrowers do incur costs beyond their collateral, in which cases the issues I have formalized in this paper become relevant, as I show in Quintin (2010). There I use my refined equilibrium concept to study the implications of recourse in mortgage markets. Specifically, I lay out a financial economy with exogenously incomplete markets where borrowers have the option to default on their financial obligations, as in DGS, and which case their liability includes their home but also, potentially, a fraction of their other assets. When all agents take lending and delivery rates as given as in DGS (2005) and optimize over quantities, many equilibria may exist making comparative statics analysis difficult. However, in this specific context, the selection procedure I outlined in this paper yields unique equilibria in all cases, making the environment amenable to studying comparative statics questions. I go on to show that when recourse becomes broader, default rates and origination yields fall on given borrower types. At the same time, the pool of borrowers changes deeply. The size of the pool can rise or fall and, likewise, average default rates can rise or fall. This application conveys one of the main messages of this paper: context-specific arguments remain necessary to establish existence in general equilibrium environments where default is a possibility.

Finally, the arguments I employ in this paper to show that endogenous default can make existence problematic are obviously related to the seminal work of Stiglitz and Weiss (1981), the paper that initially motivated the conventional belief that default and disequilibrium go hand-in-hand. As in Stiglitz and Weiss (1981), the natural equilibrium that arises in this paper is one where some agents are excluded from borrowing even though they would be willing to accept higher borrowing rates. As I discuss in the final section of this paper, the intuition for the rationing outcome in this paper differ substantially from the intuition behind the findings of Stiglitz and Weiss (1981). The issue here need not be that the composition of the pool of borrowers changes adversely when rates increase. Instead, my examples rely on

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<sup>3</sup>See, e.g., Gerardi et. al. (2007), Gerardi et. al. (2009a) and Gerardi et. al. (2009b).

the intuitively obvious fact that identical borrowers become more prone to default when their financial obligations increase. Recent work (see Arnold and Riley, 2009) has argued that the possibility of rationing in the Stiglitz-Weiss framework may be fragile. This paper’s findings suggest that rationing may prevail in models of default for more robust reasons.

## 2 The DGS equilibrium concept

DGS (2005) describes a financial economy with exogenously incomplete markets where agents can choose to renege on their financial obligations. Specifically, they consider an environment with two dates – 0 and 1 – and  $S$  possible states at date 1. For simplicity, I will assume that only one commodity is traded at each date since this suffices to establish the points I wish to make and entails no loss of generality for my purposes.

The economy contains an equal mass of a finite set  $H$  of agent types with utility function  $u^h : \mathbb{R}_+^{S+1} \mapsto \mathbb{R}$  which is continuous, concave and strictly increasing for all  $h \in H$ . Letting  $i$  index agents on the unit interval, order agents so that agent  $i \in (\frac{h-1}{H}, \frac{h}{H}]$  is of type  $h \in \{1, 2, \dots, H\}$ . For expositional simplicity, DGS (2005) focus on the case where the number of agents is finite, but doing so without compromising existence requires imposing stringent restrictions on punishment technologies such that the each agent’s problem remains convex. As DGS (2005) explain,<sup>4</sup> an existence argument that holds only for a linear specification of default punishment would be of little interest or generality. In this paper, I need to work with a broader class of punishment.

Agents of type  $h \in H$  are endowed with  $e^h \in \mathbb{R}^{S+1}$ . Agents trade the commodity at each state  $s \in \{0\} \cup S$  at a spot price I normalize to be  $p_s \equiv 1$  in all states.<sup>5</sup> Agents also trade a

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<sup>4</sup>See page 9: “One could easily imagine a legal system that imposes penalties that are nonconcave and even discontinuous in the size of the default, for example, trigger penalties that jump to a minimum level at the first infinitesimal default. One could also imagine confiscation of commodities in case of default. Our model does not explicitly allow for these possibilities. But as we show in our working paper (Dubey, Geanakoplos, and Shubik (2000)), with a continuum of households, such modifications to the default penalties do not destroy the existence of equilibrium.”

<sup>5</sup>This is the notational simplification the one-commodity assumption makes possible.

set  $J$  of assets and  $R_j \in \mathbb{R}^S$  denotes the quantity of the commodity to be delivered by asset  $j$  at each possible state at date 1, while  $\pi_j$  denotes the price of asset  $J$  at date 0. Agents of type  $h \in H$  face a short-sale constraint  $Q_j^h$  on asset  $j \in J$ .

As they would in standard financial economies, agents choose a consumption vector  $x \in \mathbb{R}_+^{S+1}$ , asset purchases  $\theta \in \mathbb{R}_+^J$ , and asset sales  $\psi \in \mathbb{R}_+^J$  taking all prices as given. However, they must also choose what quantities  $D \in \mathbb{R}_+^{S \times J}$  to deliver on each asset at each possible state at date 1. In particular, an agent can choose to deliver an amount on asset  $j$  and date  $s$  that is lower than the amount  $\psi_j R_{sj}$  they owe. If an agent chooses to deliver

$$D_{sj} < \psi_j R_{sj},$$

then the agent is defaulting on his/her promise on asset  $j$  in state  $h$ .

Agents of type  $h \in H$  who default experience a default cost measured in utility terms and equal to  $g^h(\psi, D; R)$  where  $g$  rises with  $\psi$  and falls with  $D$ . As discussed above, while DGS (2005) focus on the case where  $g^h$  is linear, their existence result can easily be extended to the general case with discontinuous or non-concave technologies provided the economy comprises a continuum of households or households can resort to a randomization device.

Asset buyers realize that they may not be paid in full at date 1. The heart of the DGS model, in fact, is a set of assumptions about expected delivery rates. First, buyers take delivery rates as given. For instance, they cannot attempt to lower asset returns in order to boost delivery rates. Neither can they opt to trade only with types that deliver at higher rates. In fact, returns are pooled and all buyers receive a pro-rata share of total deliveries on a given asset.

Given expected delivery rates  $K \in [0, 1]^{S \times J}$  and given asset prices  $\pi \in \mathbb{R}_+^J$ , the budget set  $B^h(\pi, K)$  of an agent of type  $h \in H$  is:

$$B^h(\pi, K) = \left\{ \begin{aligned} (x, \theta, \psi, D) &\in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{J \times S} \\ x_0 - e_0^h + \pi \cdot (\theta - \psi) &\leq 0 \\ \psi_j &\leq Q_j^h \text{ for } j \in J \\ x_s - e_s^h + \sum_{j \in J} D_{sj} &\leq \sum_{j \in J} \theta_j K_{sj} \cdot R_{sj} \end{aligned} \right\}.$$

A DGS equilibrium in this context is a list  $(\pi, K, \{x^i, \theta^i, \psi^i, D^i\}_{i \in [0,1]})$  such that:

1. For all  $h \in H$  and almost all  $i \in (\frac{h-1}{H}, \frac{h}{H}]$ ,

$$(x^i, \theta^i, \psi^i, D^i) \in \arg \max_{B^h(\pi, K)} u^h(x^i) - g^h(\psi, D; R),$$

2.  $\int_i (x^i - e^i) di = 0,$

3.  $\int_i (\theta^i - \psi^i) di = 0,$

4.  $K_{sj} = \frac{\int_i D_{sj}^i di}{\int_i R_{sj} \psi_j^i di}$  if  $\int_i R_{sj} \psi_j^i di > 0$  for all  $(s, j) \in S \times J$ .

The only non-standard aspect of this definition is the final condition which states that the beliefs agents form about delivery rates prior to choosing their consumption and asset holding plans must be borne out in equilibrium. It should be evident that degenerate equilibria supported by beliefs so pessimistic that no asset is actively traded always exist. DGS (2000, 2005) provide a simple trembling-hand refinement that rules out degenerate equilibria of that sort, and show that an equilibrium, in this refined set, must exist in this environment.

While DGS (2000, 2005) rule out equilibria where asset markets are shut down by excessive pessimism, beliefs continue to matter critically in equilibrium. Intuitively, buyers who anticipate low delivery rates are likely to require low asset prices to be willing to participate in a given asset market and, in turn, low asset prices could cause low delivery rate. This creates an environment propitious to multiple equilibria. In fact, I will now show by way of example



that equilibria exist where, holding all other prices and quantities traded the same, raising the price of an asset – lowering the stated return, that is – can increase delivery rates and effective rates of returns. I will argue that these equilibria are incompatible with price-taking in equilibrium. By raising the price at which they purchase assets, buyers could raise their payoffs. Put another way, by lowering the stated returns they require from borrowers, lenders can raise their effective returns.

### 3 Return-dominated equilibria

Consider a version of the economy described above with two agents types ( $H = 2$ ) and only one state at date 1 ( $S = 1$ ). There is one asset ( $J = 1$ ) with payoff  $R = 1$  at date 1. Agents of type 1 are endowed with  $e^1 = (1, 0)$  (one unit of the commodity at date 0, and zero at date 1), while agents of type 2 are endowed with  $e^2 = (0, B)$  where  $B > 0$ .

Agents of type 1 only care about consumption at date 1.<sup>6</sup> As a result, these agents always save their endowment at date 0, and there is no need to define  $g^1$  precisely. Furthermore, letting  $\theta^1$  denote their holdings of the asset, any equilibrium in this environment must feature  $\theta^1 = \frac{1}{\pi_1}$  where  $\pi_1 > 0$  is the price of the one asset.

Given  $\pi_1$  and an anticipated delivery rate  $K \in [0, 1]$  on the one asset, agents of type 2 choose  $(x_0, x_1, \theta, \psi, D)$  to maximize:

$$A \min \{x_0, 1\} + x_1 - \frac{\lambda}{2} (\psi R - \eta D)^2 - \tau 1_{\{D < \psi R\}}$$

subject to:

$$\begin{aligned} x_0 + \theta \pi_1 - \psi \pi_1 &= 0 \\ x_1 + D &= B + \theta K R \end{aligned}$$

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<sup>6</sup>Strictly speaking, as DGS (2005) explain in their footnote 20, this violates the letter of their strict monotonicity assumption, but their existence theorem is unaffected by this deviation. Alternatively, it should be clear that the example only requires that agents of type 1 discount consumption at date zero at a sufficiently high rate.

where  $A > 0$ ,  $\lambda > 1$ ,  $\tau > 0$ , and  $\eta > 1$ . Notice that the punishment technology features a fixed default cost ( $\tau > 0$ ) which will play an important role in this example. Agents of type 2 will only default provided the associated savings are high enough which only occurs – as I will now argue – provided  $\pi_1$  is low enough so that the effective return on the asset is high.

It should be clear that as long as  $A$  is high enough, agents of type 2 have no reason to purchase the asset, so that  $\theta^2 = 0$ . Then, any equilibrium in this example must feature  $\psi^2 = \frac{1}{\pi_1}$ . Again, as long as  $A$  is high enough, it is in fact optimal for agents of type 2 to set  $x_0 = 1$  and finance this consumption by selling quantity  $\frac{1}{\pi_1}$  of the asset.<sup>7</sup> Then, agents of type 2 need only decide whether or not to default on their obligations at date 2 and, when they choose to default, how much to deliver.

Standard manipulations of first-order conditions<sup>8</sup> show that if they default, it is optimal for agents of type 2 to select:

$$D = \frac{1}{\eta} \left( \psi R - \frac{1}{\lambda \eta} \right) = \frac{1}{\eta} \left( \frac{1}{\pi_1} - \frac{1}{\lambda \eta} \right)$$

Note that  $D < \frac{1}{\pi_1}$ , naturally, and that  $D \geq 0$  provided  $\frac{1}{\pi_1} > \frac{1}{\lambda \eta}$ . When  $D > 0$ , the punishment penalty equals

$$\tau + \frac{\lambda}{2} \left( \frac{1}{\lambda \eta} \right)^2,$$

while overall utility is:

$$A + \left( B - \frac{1}{\eta} \left( \frac{1}{\pi_1} - \frac{1}{\lambda \eta} \right) \right) - \tau - \frac{\lambda}{2} \left( \frac{1}{\lambda \eta} \right)^2. \quad (3.1)$$

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<sup>7</sup>This is where capping utility at 1 for agents of type 2 simplifies the example.

<sup>8</sup>Conditional on defaulting, agents maximize an objective function that is strictly concave in  $D$ . Provided the optimal delivery level is interior, the first-order condition for the optimal delivery choice is:

$$\begin{aligned} -1 + \lambda \eta \left( \frac{1}{\pi_1} - \eta D \right) &= 0 \\ \iff D &= \frac{1}{\eta} \left( \frac{1}{\pi_1} - \frac{\eta}{\lambda} \right). \end{aligned}$$

If, on the other hand, agents choose to deliver on all their promises then they incur no punishment and utility is

$$A + B - \frac{1}{\pi_1} \quad (3.2)$$

Agents default when (3.1) > (3.2) i.e. when:

$$\frac{1}{\pi_1} \left(1 - \frac{1}{\eta}\right) > \frac{\lambda}{2} \left(\frac{1}{\lambda\eta}\right)^2 - \frac{1}{\lambda\eta^2} + \tau$$

Since  $\eta > 1$ , the left-hand side of the inequality above rises with  $\frac{1}{\pi_1}$ . Let  $\pi_1^*$  be such that the default condition holds as an equality, and assume for simplicity that parameters are such that  $\frac{1}{\pi_1^*} > \frac{1}{\lambda\eta}$ , which can be guaranteed for instance by making  $\tau$  large enough.

Then, this economy generate a continuum of DGS equilibria indexed by  $\pi_1 > 0$ . When  $\frac{1}{\pi_1} < \frac{1}{\pi_1^*}$ , agents of type 2 deliver in full in period 1. When  $\frac{1}{\pi_1} = \frac{1}{\pi_1^*}$ , agents of type 2 are indifferent between delivering  $\bar{D} \equiv \frac{1}{\pi_1^*}$  and delivering  $\underline{D} = \frac{1}{\eta} \left(\frac{1}{\pi_1^*} - \frac{1}{\lambda\eta}\right)$ . Assigning various masses of type 2 agents to those two choices can produce any equilibrium at that asset price with  $K \in [\underline{D}\pi_1^*, 1]$ . Finally, when  $\frac{1}{\pi_1} \in \left(\frac{1}{\pi_1^*}, B\right]$ , agents deliver  $\frac{1}{\eta} \left(\frac{1}{\pi_1} - \frac{1}{\lambda\eta}\right)$ .

These equilibria are depicted in figure 1. When the asset return reaches  $\frac{1}{\pi_1^*}$ , agents of type 2 begin defaulting hence the amount received by agents of type 1 falls. Only when the return becomes large enough, at a level I denote by  $\frac{1}{\pi_1^{**}}$  on the figure, do delivery rates reach the same level as at  $\frac{1}{\pi_1^*}$ . Equilibria in the  $\left(\frac{1}{\pi_1^*}, \frac{1}{\pi_1^{**}}\right]$  interval are pathological because lenders (agents of type 1) could choose to lower the returns they demand on the assets they purchase, which agents of type 2 would welcome, and increase delivery rates.

One aspect of this example that may seem important at first glance is the fact that agents of type 2 only face one type of agents hence do not have to worry about selection issues. However, this plays no role in generating return-dominated equilibria. Recognizing and taking advantage of the profit opportunity that exists when  $\frac{1}{\pi_1} \in \left(\frac{1}{\pi_1^*}, \frac{1}{\pi_1^{**}}\right]$  does not require that agents of type 2 know the identity or type of the agents with whom they are dealing. It only requires that they understand the structure of the economy in which they are lending. A pool of agents of type 2 (or an intermediary representing them) who would offer a lower

interest rate than the equilibrium rate would attract the interest of all agents of type 1, and any random selection device would yield a representative set of agents of that type. Anonymity, in other words, cannot possibly preclude lenders from lowering stated returns in order to increase effective returns.

Assuming that lenders face a continuum of borrower types is useful in fact if one wishes to generate a hill-shaped relationship between stated returns and effective returns as drawn by Stiglitz and Weiss (1981) instead of the stark break displayed in figure 1, using the fact that different borrowers would then default at different thresholds. Quintin (2011) provides a specific example. Arnold and Riley (2009) argue that the textbook hill-shape case is not a robust outcome in the model of Stiglitz and Weiss (1981). That issue does not arise in the DGS model.

All told then, the DGS concept produces equilibria where lenders ignore a glaring profit opportunity. This possibility makes the DGS framework inconsistent with assuming price-taking behavior on the part of lenders. Equilibria such as those depicted in figure 1 could not possibly persist in the economic environment described by DGS.

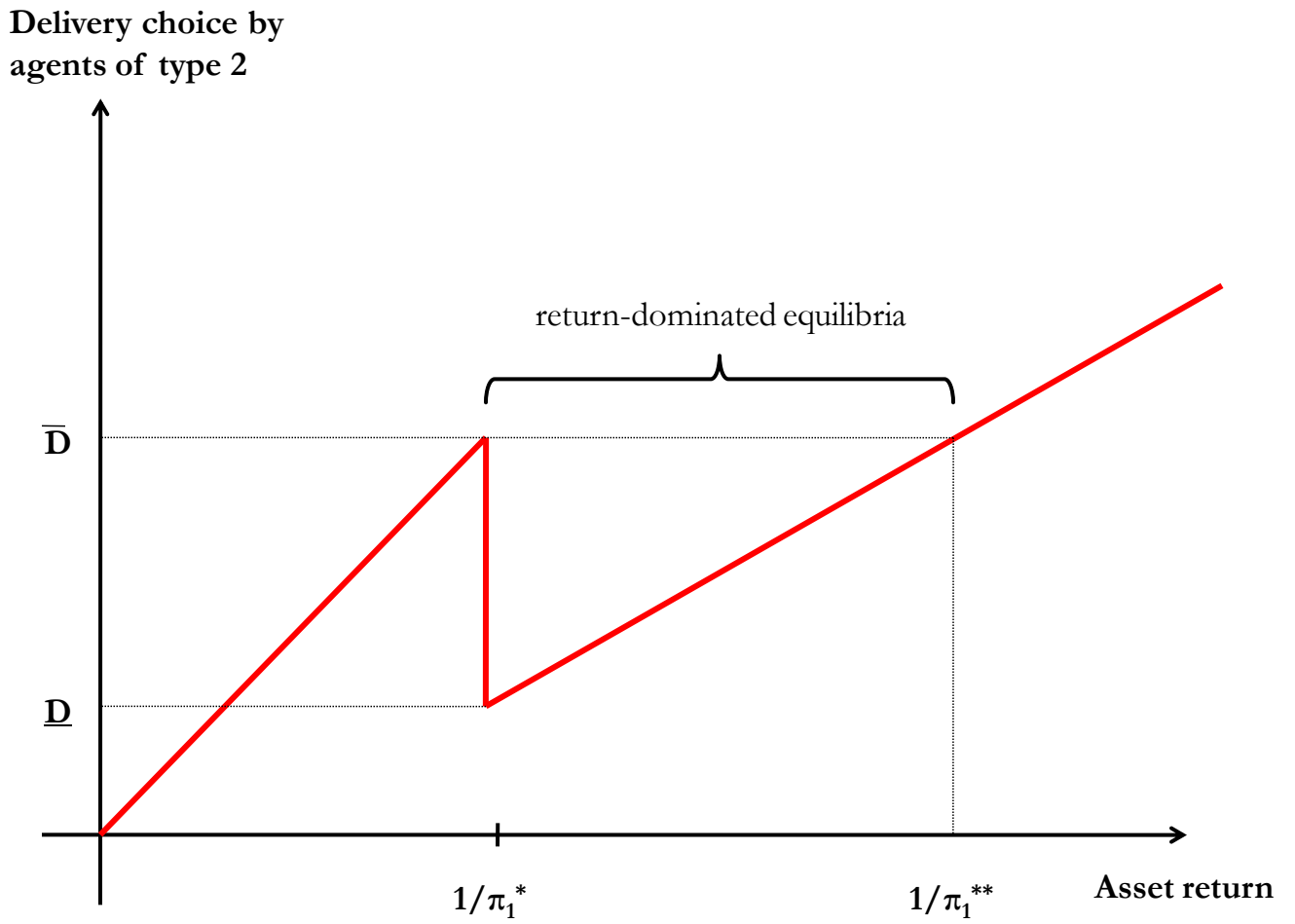
## 4 A selection procedure

One way to rule out the class of equilibria described in the previous section is to relax the assumption that asset buyers take prices as given and make asset price decisions choice variables. Implementing this solution requires a drastic departure from the DGS model. Another approach, which I pursue here, is to select equilibria in the DGS set that give buyers no incentives to depart from equilibrium prices.

First, we need a systematic way to describe the equilibria we seek to rule out. To that end, it is useful to introduce some notation. For  $(\pi, K) \in \mathbb{R}_+^J \times \mathbb{R}_+^{S \times J}$ , define

$$\mathcal{K}(\pi, K) = \left\{ \left( \frac{\int_i D_{sj}^i di}{\int_i R_{sj} \psi_j^i di} : s \in S, j \in J \right) : D^i \text{ solves agent } i\text{'s problem given } (\pi, K) \right\},$$

Figure 1: A continuum of equilibria



with the convention that the delivery rate is set to zero when the denominator of the integral is zero.

Note that in equilibrium, consistency requires that  $K \in \mathcal{K}(\pi, K)$ . In general,  $\mathcal{K}$  lists all the delivery rates compatible with optimal behavior on the part of agents given  $(\pi, K)$ . Standard arguments show that  $\mathcal{K}$  is non-empty and convex valued, and that it has a closed graph. I can now state:

**Definition 4.1.** *A DGS equilibrium  $(\pi, K, \{x^i, \theta^i, \psi^i, D^i\}_{i \in [0,1]})$  is return-dominated if there exists  $\hat{\pi} \in \mathbb{R}_+^J$  such that:*

1.  $\hat{\pi}_{j^*} > \pi_{j^*}$  for some  $j^* \in J$  such that  $\int \psi_{j^*}^i > 0$  while  $\hat{\pi}_j = \pi_j$  if  $j \neq j^*$  and,
2. there exists  $\hat{K} \in \mathcal{K}(\hat{\pi}, K)$  such that  $\frac{\hat{K}_{sj^*} R_{sj^*}}{\hat{\pi}_{j^*}} > \frac{K_{sj^*} R_{sj^*}}{\pi_{j^*}}$  for all  $s \in S$ .

In such an equilibrium, a small (measure zero) set of agents who are buying asset  $j^*$  could lower the return they require on that asset, select a small (measure zero), representative set of agents currently selling the asset with whom to trade, and strictly raise their income in the second period by offering these borrowers terms they strictly prefer. Since all buyers of asset  $j^*$  should recognize this profit opportunity and no friction in the DGS environment precludes them from taking advantage of it, a return-dominated equilibrium could not persist in a competitive world.

This argument presumes that asset  $j^*$  in definition 4.1 remains sold by borrowers once its price rises. That this has to be the case is immediate since the delivery rate is zero by convention when nobody sells the asset. Since delivery rates rise strictly at the counterproposal price vector  $\hat{\pi}$ , they must be positive. But even absent this convention, a more fundamental argument can be used to demonstrate that asset  $j^*$  must remain sold by borrowers following the price change. To see this, take any agent  $i$  for whom  $\psi_{j^*}^i > 0$  at the original equilibrium. When the price of asset  $j^*$  rises, their welfare must rise strictly. Indeed, these agents could simply choose to increase their consumption at date 0 and leave all other plans unchanged. So assume by way of contradiction that following the price change, these agents choose to set

$\psi_{j^*}^i = 0$ . Since  $0 \times \pi_{j^*} = 0 \times \hat{\pi}_{j^*}$  the new plan was feasible at the original price vector hence cannot raise the welfare of these agents, which is the contradiction we sought.

To illustrate the exclusion procedure further, notice that in figure 1, the equilibria that meet the two criteria stated in definition 4.1 are precisely the equilibria I called incompatible with price taking on the price of lenders in the previous section. To see this, denote by  $D(\pi_1)$  the delivery level chosen by agents of type 2 when the equilibrium price is  $\pi_1$ , making their delivery rate  $K(\pi_1) = \frac{D(\pi_1)}{\frac{1}{\pi_1} \times 1}$  since in equilibrium agents of type 2 sell quantity  $\frac{1}{\pi_1}$  of the asset.

Then, an equilibrium at price  $\pi_1$  is return-dominated according to my criterion if another price  $\pi'_1$  exists such that

$$\pi'_1 > \pi_1 \tag{4.1}$$

and:

$$\begin{aligned} \frac{K(\pi'_1)}{\pi'_1} &> \frac{K(\pi_1)}{\pi_1} \\ \frac{\frac{D(\pi'_1)}{\frac{1}{\pi'_1}}}{\pi'_1} &> \frac{\frac{D(\pi_1)}{\frac{1}{\pi_1}}}{\pi_1} \\ \iff \frac{D(\pi'_1)}{\pi'_1} &> \frac{D(\pi_1)}{\pi_1} \\ \iff D(\pi'_1) &> D(\pi_1) \end{aligned} \tag{4.2}$$

The first line of this string of inequalities is my return-domination criterion in this specific case since I normalize the return on the asset to 1 so that delivery per unit of asset bought when the price is  $\pi_1$  is simply  $K(\pi_1)$ . Therefore, an equilibrium at price  $\pi_1$  is return-dominated and excluded under my procedure if and only if both condition (4.1) and (4.2) hold. On figure 1, these are the equilibria in the interval  $(\frac{1}{\pi_1^*}, \frac{1}{\pi_1^{**}})$ , as claimed.

The most natural way to deal with this issue seems to be selecting equilibria that are not return-dominated. The question is whether there is such an equilibrium in the DGS set and I will now show that the answer, in general, may be negative.

## 5 Existence

The main point of DGS (2005) is that default is not incompatible with the “orderly functioning of markets” in the sense that introducing a default option in an otherwise standard general equilibrium model with incomplete markets does not jeopardize existence. I have argued that the DGS concept may generate equilibria that hardly seem compatible with price taking on the part of lenders. A natural question is whether there always exists at least one equilibrium that is not return-dominated.

Notice that on figure 1, there is a continuum of equilibria that are not return dominated and these happen to coincide with the set of DGS equilibria that are constrained efficient in the sense of Geanakoplos and Polemarchakis (1986). These equilibria are such that no reallocation of asset holdings alone could improve every agent’s utility. I will now show that constrained-efficiency is in fact sufficient for an equilibrium to survive my exclusion procedure in the special case where  $S = 1$ . Specifically, the following result establishes the fact that constrained efficient equilibria are not return-dominated in that case.<sup>9</sup>

**Remark 5.1.** *Return-dominated DGS equilibria are constrained-inefficient when  $S = 1$ .*

*Proof.* Take a DGS equilibrium  $(\pi, K, \{x^i, \theta^i, \psi^i, D^i\}_{i \in [0,1]})$  that is return-dominated and let  $j^* \in J$  be an asset on which it is possible to raise effective returns by lowering stated returns. Since delivery rates are bounded above by 1, for  $q > 0$  high enough,  $\frac{\bar{K}_{j^*} R_{j^*}}{\bar{\pi}_{j^*}} < \frac{K_{j^*} R_{j^*}}{\pi_{j^*}}$  for any  $\bar{K} \in \mathcal{K}(\bar{\pi}, K)$  where  $\bar{\pi}_j = \pi_j$  for  $j \neq j^*$  while  $\bar{\pi}_{j^*} = q$  if  $j = j^*$ . Since  $\mathcal{K}$  is non-empty and convex valued and has a closed graph, it follows that there exist  $\tilde{\pi} \in \mathbb{R}_J^+$  such that  $\tilde{\pi} \geq \pi$  with a strict inequality only at coordinate  $j^*$  such that  $\frac{\tilde{K}_{j^*} R_{j^*}}{\tilde{\pi}_{j^*}} = \frac{K_{j^*} R_{j^*}}{\pi_{j^*}}$  for some  $\tilde{K} \in \mathcal{K}(\tilde{\pi}, K)$ . At this new set of prices, it is budget feasible for all agents to choose the same consumption and delivery plan as before by setting  $\tilde{\theta}_{j^*}^i \equiv \theta_{j^*}^i \times \frac{\pi_{j^*}}{\tilde{\pi}_{j^*}}$  and  $\tilde{\psi}_{j^*}^i \equiv \psi_{j^*}^i \times \frac{\pi_{j^*}}{\tilde{\pi}_{j^*}}$ . At this new trading plan, however, the punishment for agents that underdeliver on asset  $j^*$  falls strictly, while

<sup>9</sup>The result is trivial when  $J = 1$  as well, but since punishment and short-sale constraints are individual and asset specific, several different assets may be traded even when  $S = 1$ .



other agents are as well off as in the original equilibrium. This is the constrained Pareto improvement we sought.  $\square$

As the proof makes clear, return-dominated equilibria are inefficient in this case in an obvious sense. The exact same consumption allocation for all agents and the same delivery plans could be supported at asset prices and assets holdings that make punishment less severe on defaulting borrowers. All agents need to do is contract at a lower stated rate of return on some assets and deliver as before, which lowers the intensity of punishment, without altering anyone's consumption plans.<sup>10</sup>

DGS equilibria that are not return-dominated must therefore exist whenever efficient equilibria exist when  $S = 1$ . More generally however, existence could fail. For one thing, it is known that even in the case without default (see Geanakoplos and Polemarchakis, 1986), equilibria are generically suboptimal when there are several goods and/or several assets. What's more, a glance at the proof above should make it clear that the return-dominated equilibria need no longer be inefficient when  $S > 1$ .

Much more specifically, a simple variation on the example I built in the previous section shows that DGS economies may generate nothing but return-dominated equilibria. To see this, take an economy populated by the same two types of agents as in the previous section, but now add a positive mass of a third type of agent ( $h = 3$ ) who never default ( $g^3 = +\infty$ , whenever  $D < \psi R$ ), have no endowment in the first period, and have a positive endowment in the second period. These agents have linear preference with discount rate  $\beta > 0$  between the two periods. Then we must have  $\frac{1}{\pi_1} > \beta$  since otherwise these agents would sell the asset which would cause supply of the asset to exceed demand since the other two types of agents have preferences such that their combined net demand for the asset is zero regardless of the price. One can then choose  $\beta \in (\frac{1}{\pi_1^*}, \frac{1}{\pi_1^{**}})$  which lops off all equilibria to the left of the default threshold.

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<sup>10</sup>While this result applies only to a rather special case, it conveys an important message. I am about to prove that universal existence arguments cannot be produced when default is a possibility and lenders behave competitively. That is not to say that existence arguments cannot be produced in models of default. Instead, the message is that existence arguments must be context-specific.

Next, assume that punishment is capped above for agents of type 2 at some upper-level  $\bar{P} > \frac{1}{\pi_1^*}$ . Agents of type 2 can now opt to deliver nothing and take that maximum punishment leaving them with utility  $A + B - \bar{P}$ . Since  $\bar{P} > \frac{1}{\pi_1^*}$ , they are better off making full delivery if the asset return is below  $\frac{1}{\pi_1^*}$ , just like in the economy described in the previous section. Assume now however that

$$A + \left( B - \frac{1}{\eta} \left( \frac{1}{\pi_1^{**}} - \frac{1}{\lambda\eta} \right) \right) - \tau - \frac{\lambda}{2} \left( \frac{1}{\lambda\eta} \right)^2 < A + B - \bar{P}.$$

That is, assume that at required return  $\frac{1}{\pi_1^{**}}$ , agents of type 2 are strictly better off making no delivery on the asset. Then, there is a threshold return in  $(\frac{1}{\pi_1^*}, \frac{1}{\pi_1^{**}})$  past which agents of type 2 become better off opting for zero delivery. We can in fact choose that threshold in  $(\beta, \frac{1}{\pi_1^{**}})$  which caps the maximum delivery level past  $\frac{1}{\pi_1^*}$  to a level  $D_{max}$  strictly below  $\bar{D}$ .

The construction is depicted in figure 2. The dotted line traces the equilibria from the previous section that become ruled out because of the presence of the new agents and because punishment is capped above. The solid line shows the DGS equilibria that remain. At any of those equilibria, lenders would be better off charging, say,  $\frac{1}{\pi_1^*}$  on their loan and getting full delivery, making all borrowers better off as well. No friction in the model could explain why lenders choose to forego this profit opportunity.

Naturally, if all lenders deviate to this new price, there is excess supply of the asset as agents of type 3 now want to sell it, so that assets markets can no longer clear. In other words, no equilibrium exists in this example other than return-dominated equilibria.

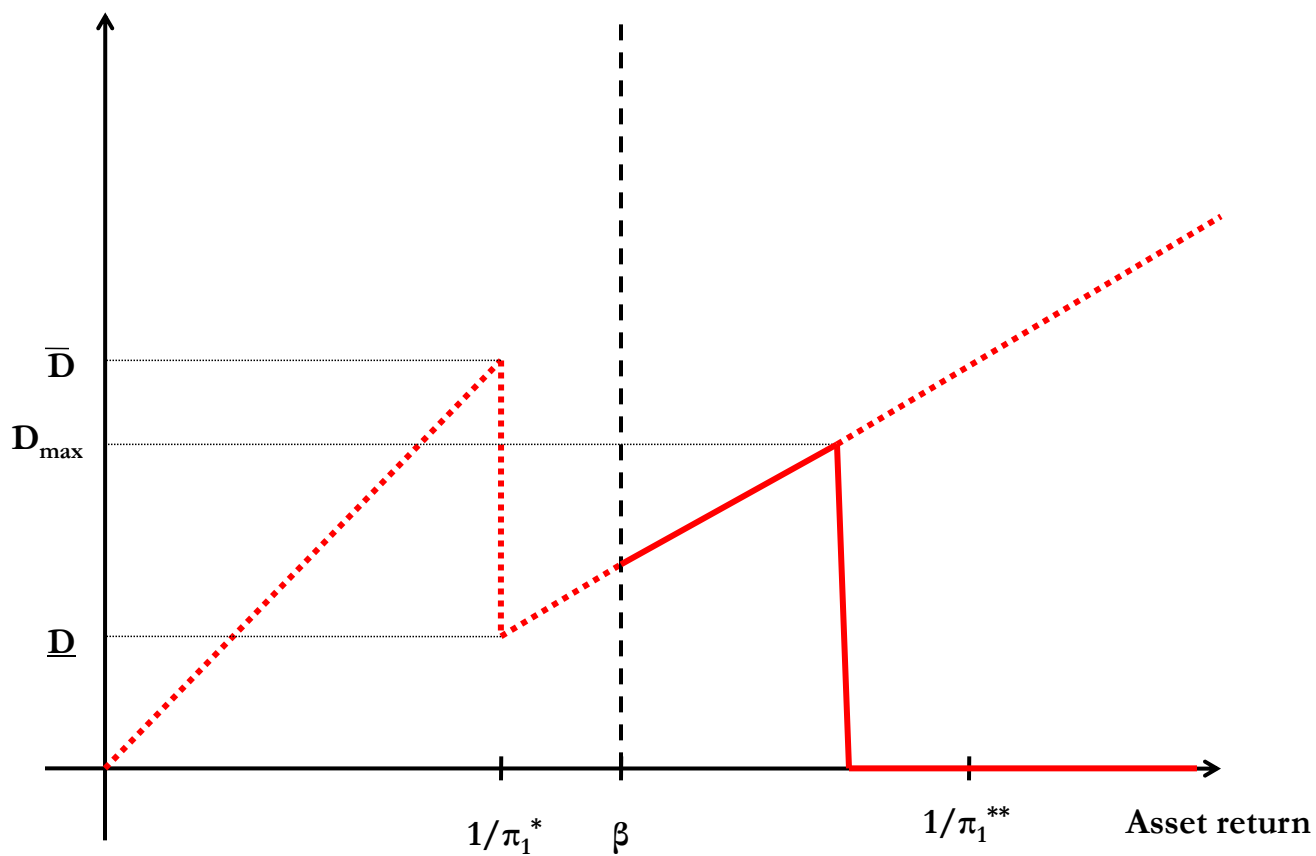
## 6 Discussion

What outcome should we expect to observe in the economy described above? One clearly stable outcome has lenders charge  $\frac{1}{\pi_1^*}$ , which maximizes delivery per unit of the asset purchased.<sup>11</sup> At that rate however, there is excess supply of the asset as agents of type 3 are

<sup>11</sup>Strictly speaking at  $\frac{1}{\pi_1^*}$ , delivery could be anywhere in the  $[\underline{D}, \bar{D}]$  interval but for simplicity I focus on the equilibrium where all borrowers choose to deliver in full at that point, as is weakly optimal for them.

Figure 2: An economy where all DGS equilibria are return-dominated

Delivery choice by agents of type 2



strictly better off selling it. Note in addition that all potential asset sellers would be willing to pay a higher rate to asset buyers. However, asset buyers, as discussed above, would anticipate the adverse consequences of so doing on expected delivery rates.

A natural outcome in this environment, therefore, is credit rationing in the sense of Stiglitz and Weiss (1981). It is important to recognize, however, that the economics of my example are very different from those that underly Stiglitz and Weiss' example. At their rationing equilibrium, raising rates reduces lenders expected returns by changing the composition of the pool of borrowers. In my example, the pool of potential borrowers does not change at all: all agents of type 2 or 3 continue to want to borrow even if rates rise.<sup>12</sup> What happens is that the delivery behavior of existing or potential borrowers changes drastically. Type 2 borrowers, specifically, choose to default following the rate increase when they chose to deliver in full before. The economics of this example are thus both trivial and compelling: raising payments increases the likelihood of default on the part of any given borrower by making the benefits of default higher. It is possible in fact and as happens in this example that this likelihood increases so much as to lower net returns.

Next, what do those examples imply for how one should approach the study of environments with endogenous default? They imply quite simply that universal existence arguments – such as those commonly invoked in classical general equilibrium environments – cannot be found in environments where asset payoffs depend on endogenous default decisions. What this means is that existence arguments when default is a possibility (the canonical case in financial applications) must be context-specific.

One example of this is Quintin (2011). There I apply my refined equilibrium concept to mortgage pricing. In that specific context, I show that my refinement not only remains compatible with existence of an equilibrium where all markets clear but in fact guarantees uniqueness, paving the way for meaningful comparative statics. I use this result to take on a

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Alternatively, I could resort to a limiting argument to argue that lenders could always approximate that outcome with arbitrary precision by approaching  $\frac{1}{\pi_1^*}$  from the left.

<sup>12</sup>In this case and to be precise, this is true until the rate reaches agent 3's participation threshold.

policy question of fundamental importance: would making punishment tougher in the event of mortgage default reduce average default rates? Since the recent foreclosure crisis has been particularly severe in states such as California or Arizona where recourse is severely limited, it is natural to conjecture that tougher statutes could have mitigated the crisis in those states. I find however that the theoretical implications of recourse are deeply ambiguous because changing recourse statutes can cause borrowers that are prone to default to enter mortgage markets. Simply put, knowing that they can collect more in some states can cause intermediaries to tolerate riskier borrowers. I go on to argue that meaningful econometric tests of whether recourse matters cannot rely on aggregate variables but must instead rely on detailed microeconomic information on borrowers at contract origination.

A different strand of the general equilibrium literature on default (see Kehoe and Levine, 1993) studies environments where a complete set of securities are traded but default endogenously limits the positions agents can take in those various securities. In those models, both welfare theorems hold – equilibria are constrained efficient, and constrained equilibria can be supported as equilibria with transfers – and existence holds with great generality. Therefore and in sharp contrast with the model discussed in this paper, default in those models cannot improve welfare and, in equilibrium, no contract is written where agents have an incentive to default in some state.<sup>13</sup> The points I have made in this paper pertain exclusively to models where markets are exogenously incomplete.

Finally, one should point out that the fact that equilibria with incomplete markets and endogenous defaults can lead to several lending rates compatible with a given opportunity cost of funds has been known in the literature that studies the impact of default statutes using quantitative methods. Part of that literature studies the effects of bankruptcy reform on credit and default.<sup>14</sup> Another part of that literature (see e.g. Corbae and Quintin, 2010) explores quantitatively the consequences of mortgage recourse for equilibrium foreclosure

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<sup>13</sup>At the same time, as Kehoe and Levine (2006) discuss, it is possible to implement the equilibrium allocation in a model with bankruptcy and collateral.

<sup>14</sup>See e.g. Athreya (1999), Chatterjee et al. (2007).

rates in the residential mortgage market. These papers argue that among the rates that are compatible with zero expected net profits on the part of the lender, the most favorable rate to the borrower should prevail.

Computationally, locating this equilibrium requires searching on a grid starting from a rate under which net lender profits must be negative. Traditional, faster approaches such as bisection run the risk of producing returns that are return-dominated precisely in the sense I have made precise in this paper, hence economically implausible. Furthermore, tolerating these equilibria in all their multiplicity would make asking the comparative statics questions these quantitative papers seek to address virtually impossible.

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