
ECOLE POLYTECHNIQUE
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

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November 2004

Cahier n° 2004-028

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Maxmin computation and optimal correlation in repeated games with signals

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Résumé: Le maxmin pour une certaine classe de jeux répétés à observation imparfaite est obtenu comme la solution d'un problème d'optimisation défini sur l'ensemble des distributions de probabilités sous contraintes d'entropie. Cette article offre une méthode pour résoudre un tel problème dans le cas d'un jeu à trois joueurs où chaque joueur dispose de deux actions à chaque étape.

Abstract: For a class of repeated games with imperfect monitoring, the maxmin payoff is obtained as the solution of an optimization problem defined on a set of probability distributions under entropy constraints. The present paper offers a method for solving such problems for the class of 3-player 2 by 2 games.

Mots clés : Jeu répété à observation imparfaite, maxmin, entropie, optimisation

Key Words : Repeated game with imperfect monitoring, maxmin, entropy, optimization

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Maxmin computation and optimal correlation in repeated games with signals

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November 18, 2004

Abstract

For a class of repeated games with imperfect monitoring, the max min payoff is obtained as the solution of an optimization problem defined on a set of probability distributions under entropy constraints. The present paper offers a method for solving such problems for the class of 3-player 2×2 -games.

1 Introduction

The cornerstone of the theory of repeated games is the Folk Theorem which states, under a variety of assumptions, that when the horizon of the game tends to infinity, the limit set of equilibrium payoffs is the set of feasible and individually rational payoffs.

Under perfect monitoring of actions, the individually rational level of a player is the minmax of his one-stage payoff function where his opponents play uncorrelated mixed strategies. In games with imperfect monitoring, information asymmetries about past play may create possibilities of correlation for the opponents.

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For instance, if players against i have perfect monitoring and if player i observes no signals, the opponents can exchange messages that are secret for player i and punish him to the minmax level in correlated mixed strategies. In general games with imperfect monitoring, the minmax level for a player lies between the correlated minmax and the uncorrelated minmax of the one-shot game.

Gossner and Tomala [GT04a] study the difference of forecasting abilities between a perfect observer of a stochastic process and an observer who gets imperfect signals on the same process. Building on this result, Gossner and Tomala [GT04b] consider repeated games where player i gets a signal on his opponents' action profile which does not depend on his own action. At a given stage of the game, i holds a belief on the mixed action profile used by players against him, represented by a probability distribution on the set of uncorrelated mixed action profiles. Such a distribution, Z , is called a *correlation system*.

To each correlation system corresponds an entropy variation, $\Delta H(Z)$, defined as the difference between the expected entropy of the mixed action profile of players against i and the entropy of the signal observed by i . Gossner and Tomala [GT04b] prove that the max min of the repeated game (where player i is minimizing) is the highest payoff obtained by using two correlation systems Z_1 and Z_2 with respective time frequencies λ_1 , $\lambda_2 = 1 - \lambda_1$ under the constraint that the average entropy variation is non-negative (*i.e.* $\lambda_1 \Delta H(Z_1) + \lambda_2 \Delta H(Z_2) \geq 0$). To achieve this payoff, the opponents of i start by generating signals that give little information to player i (they accumulate entropy). Then they play alternatively a correlation system that yields a bad payoff but generates entropy (has a positive entropy variation) and another that uses the entropy just generated to yield a good payoff. The constraint on the frequencies of the correlation system is that on average, the entropy variation must be greater than or equal to zero.

The aim of the present paper is to develop tools for computing optimal solutions of this problem when the team against player i consists of two play-

ers. In order to compute the combination of correlation system that yields the highest payoff under the entropy constraint, we study the auxiliary problem that consists in computing optimal correlation systems associated with a fixed correlated distribution of actions of the team: this is the correlation system with maximal entropy variation inducing this distribution. We derive general properties of the solutions and a full characterization of these solutions when each of the team player's action spaces has two elements. Relying these solutions, we deduce a full analytic characterization of the max min of an example of repeated game with imperfect monitoring. Another application of our characterization of optimal correlation systems has been developed by Goldberg [Gol03] (see section 5 below). Beyond the game studied in this paper, the tools we develop may serve as a basis for computations of solutions of maximization problems under entropy constraints raising from other optimization or game theoretic problems.

This paper is part of a growing body of literature on entropy methods in repeated games. Lehrer [Leh88] and Lehrer and Smorodinsky [LS00] use the relative entropy as a distance between probability measures. Neyman Okada [NO99], [NO00] use entropy as a measure of the randomness of a mixed strategy, and apply it to repeated games played by boundedly rational players. Gossner and Vieille [GV02] compute the max min value of a zero-sum repeated game where the maximizing player is not allowed to randomize freely but privately observes an exogenous i.i.d. process, and show that this value depends on the exogenous process through its entropy only. Gossner, Hernandez and Neyman [GHN04] apply entropy methods to the study of optimal use of communication resources.

We present the model, an example and the auxiliary optimization problem in section 2. Section 3 is devoted to optimal correlation systems. Section 4 solves the specific example. The proofs of the main results are postponed to section 5. We discuss possible extensions and applications in section 6.

2 The repeated game with signals

We consider a 3-player finite game Γ where players 1 and 2 have identical payoffs opposed to the payoff of player 3. The set of players is $I = \{1, 2, 3\}$, A^i is player i 's finite set of actions, and $A = A^1 \times A^2 \times A^3$. The payoff function to players 1 and 2 is $g : A \rightarrow \mathbb{R}$, and $-g$ is the payoff function of player 3. The set of mixed strategies for player i is $X^i = \Delta(A^i)$, and the set of correlated strategies for players 1 and 2 is $X^{12} = \Delta(A^1 \times A^2)$. For $x \in X^1$ and $y \in X^2$, $x \otimes y$ represents the direct product of x and y , and the subset of X^{12} consisting of product measures is $X = X^1 \otimes X^2$.

The game is infinitely repeated and at after each stage $t = 1, 2, \dots$, if $a = (a^1, a^2, a^3)$ is the profile of actions played by the players at stage t , both players 1 and 2 observe a while player 3 observes $f(a^1, a^2)$ where f is a fixed mapping from $A^1 \times A^2$ to some finite set of signals. Players 1 and 2 thus have perfect monitoring and player 3 has imperfect monitoring. The team $\{1, 2\}$ is maximizing the long-run average payoff while player 3 is minimizing this payoff. The solution concept we consider is the max min of the infinitely repeated game (for a precise definition see e.g. [GT04b]).

2.1 An example

For a given payoff specification, we show how the signalling structure affects the max min payoff. One of the aims of this paper is to compute the max min payoff for this payoff specification and one signalling structure of particular interest.

The payoff specification is as follows. Players 1 and 2 play a coordination game: each of them chooses between spending the evening at the bar ‘Golden Gate’ (G) or at the bar ‘Happy Hours’ (H). Player 3 faces the same choice. The payoff for the first two players is 1 if they meet at the same bar and 3 chooses the other bar, otherwise the payoff is 0. The payoff function is displayed below where 1 chooses the row, 2 the column and 3 the matrix.

	G	H
G	0	0
H	0	1
	G	

	G	H
G	1	0
H	0	0
	H	

The uncorrelated max min of the one-shot game is $\frac{1}{4}$ and may be obtained in the repeated game by the team $\{1, 2\}$ by playing the same mixed action $(\frac{1}{2}, \frac{1}{2})$ at every stage.

The correlated max min of the one-shot game is $\frac{1}{2}$. This may be obtained by players 1 and 2 in the repeated game if they can induce player 3 to believe, at almost every stage, that (G, G) and (H, H) will both be played with probability $\frac{1}{2}$ and if their play is independent on player 3's behavior. For example, if player 3 has no information concerning the past moves of the opponents, then team $\{1, 2\}$ may achieve its goal by randomizing evenly at the first stage, and coordinate all subsequent moves on the first action of player 1.

The case of particular interest is when player 3 observes the actions of player 2 but not of player 1, *i.e.* $f(a^1, a^2) = a^2$. The study of this game with this signalling structure, which we denote Γ_0 , was proposed by [RT98].

The following strategies for players 1 and 2 achieve partial correlation in the repeated game:

- At odd stages, play $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})$,
- at even stages, repeat the previous move of player 1. Player 3's belief is then that (G, G) is played with probability $\frac{1}{2}$ and (H, H) with the same probability.

The limit time-average payoff yielded by this strategy is $\frac{3}{8}$. How much correlation can be achieved by the team $\{1, 2\}$ in this game? Can the team improve on $\frac{3}{8}$? Is it possible to achieve complete correlation? As an application of our main results (theorem 7) we prove that full correlation is not achievable, but the team can improve sharply on $\frac{3}{8}$. Some optimal level of

correlation is achievable, and the corresponding payoff lies between 0.4020 and 0.4021 (corollary 10).

2.2 The optimization problem

The entropy of a finite support probability distribution $P = (p_l)_{l=1}^L$ is given by $H(P) = -\sum_{l=1}^L p_l \log(p_l)$ where $\log = \log_2$ and $0 \log 0 = 0$. For the case of binary distributions $(x, 1-x)$, we let $h(x) = -x \log(x) - (1-x) \log(1-x)$.

Definition 1 *A correlation system Z is a distribution with finite support on X :*

$$Z = \sum_{k=1}^K p_k \delta_{x_k \otimes y_k}$$

where for each k , $p_k \geq 0$, $\sum_k p_k = 1$, $x_k \in X^1$, $y_k \in X^2$ and $\delta_{x_k \otimes y_k}$ stands for the Dirac measure on $x_k \otimes y_k$. Hence, for each k , the probability under Z of $x_k \otimes y_k$ is p_k .

- The **distribution of actions** for players 1 and 2 induced by Z is $D(Z) = \sum_k p_k x_k \otimes y_k$, an element of X^{12} .
- The **payoff** yielded by Z is $\pi(D(Z)) = \min_{a^3} g(D(Z), a^3)$.
- The **distribution of signals** for player 3 induced by Z is $f(D(Z))$, i.e. for each signal u , $f(D(Z))(u) = \sum_{f(a^1, a^2)=u} D(Z)(a^1, a^2)$.
- The **entropy variation** of Z is $\Delta H(Z) = \sum_k p_k (H(x_k) + H(y_k)) - H(f(D(Z)))$.

The interpretation is the following. The set $\{1, \dots, K\}$ represents the set of events secretly observed by the team before a given stage, and p_k is the probability that player 3 assigns to the event k . Under Z , players (1, 2) play (x_k, y_k) if they observe k . Hence, given player 3's information (who ignores the true value of k) the distribution of actions of the team is $D(Z)$. Player 3 plays a best response to $D(Z)$ and the corresponding payoff is $\pi(D(Z))$. The amount $\sum_k p_k (H(x_k) + H(y_k))$ is the expected entropy of

the mixed profile of the team, whereas $H(f(D(Z)))$ represents the entropy of the signal observed by player 3. The difference between the entropy added to the information of the team and of the new information to player 3 is the entropy variation $\Delta H(Z)$.

Examples. Consider the game Γ_0 . We identify a mixed strategy x for player 1 [resp. y for player 2] to the probability it puts on G .

- Let $Z = \delta_{x \otimes y}$. The payoff $\pi(D(Z))$ is then $\min\{xy; (1-x)(1-y)\}$. Since the signal is the move of player 2, $f(D(Z))$ puts weight y on G . Applying definition 1, we have $\Delta H(Z) = (H(x) + H(y)) - H(y) = H(x)$. Among the set of correlation systems that are Dirac measures, the payoff is maximal for $x = y = \frac{1}{2}$ and takes the value $\frac{1}{4}$. The entropy variation is also maximal for $x = y = \frac{1}{2}$ and takes the value 1. Let $Z_{\frac{1}{2}} = \delta_{\frac{1}{2} \otimes \frac{1}{2}}$.

- Let $Z_1 = \frac{1}{2}\delta_{1 \otimes 1} + \frac{1}{2}\delta_{0 \otimes 0}$. The distribution induced is $D(Z_1) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and the associated payoff is $\pi(D(Z_1)) = \frac{1}{2}$. The distribution of signals $f(D(Z))$ puts weight $\frac{1}{2}$ on both G and H and for each k , $h(x_k) = h(y_k) = 0$, so the entropy variation is -1 .
- The cyclic strategy devised in case 3 of section 2.1 consists in playing $Z_{\frac{1}{2}}$ at odd stages and Z_1 at even stages, so that we cyclically gain and lose 1 bit of entropy. The payoff obtained is the average between these of $Z_{\frac{1}{2}}$ and Z_1 , hence $\frac{3}{8}$.

Consider the map $U: \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ defined by the following optimization problem:

$$U(c) = \sup_{Z: \Delta H(Z) \geq c} \pi(D(Z))$$

We recall the following result from Gossner and Tomala [GT04b]

Lemma 2 *The max min of the infinitely repeated game is*

$$\text{cav } U(0)$$

where $\text{cav} U$ is the smallest concave function pointwise greater than U .

The max min of the repeated game with imperfect monitoring thus writes as the maxmin of a game in which the team $\{1, 2\}$ may choose an arbitrary correlation system at each stage under the constraint that the average entropy variation is non-negative. This represents the optimal trade-off for this team between generating correlation (by playing some Z_1 with $\Delta H(Z_1) > 0$), and using correlation to get a good payoff (by playing some Z_2 with $\Delta H(Z_2) < 0$).

3 An auxiliary optimization problem

We develop the analytic tools that facilitate the computation of the map U . We do this through the resolution of an auxiliary optimization problem.

3.1 Optimal correlation systems

Since the payoff $\pi(D(Z))$ depends on Z through the induced distribution $D(Z)$ only, we study how to induce a given distribution D by a Z with maximal entropy variation.

Definition 3 Given $D \in X^{12}$, a correlation system Z is **optimal for D** if:

1. $D(Z) = D$;
2. For every Z' such that $D(Z') = D$, $\Delta H(Z') \leq \Delta H(Z)$.

A correlation system Z is **optimal** if it is optimal for $D(Z)$.

The existence of optimal correlation systems does not follow directly from the definition, but is a consequence of the following proposition.

Proposition 4 For every $D \in X^{12}$, there exists Z optimal for D which has finite support of cardinal no more than $|A^1| + |A^2|$.

Proof. Let $D \in X^{12}$, identifying an action a^i of player i with the mixed strategy $\delta_{a^i} \in X^i$, one has:

$$D = \sum_{a^1, a^2} D(a^1, a^2) \delta_{a^1 \otimes a^2}$$

Thus the set of Z such that $D(Z) = D$ is non-empty. Now for each $Z = \sum_{k=1}^K p_k \delta_{x_k \otimes y_k}$ such that $D(Z) = D$, the vector $(D(Z), \Delta H(Z))$ writes:

$$(D(Z), \Delta H(Z)) = \sum_{k=1}^K p_k (\delta_{x_k \otimes y_k}, H(x_k) + H(y_k) - H(f(D)))$$

and, indentifying $\delta_{x_k \otimes y_k}$ with $x_k \otimes y_k$, it belongs to the convex hull of the set:

$$S = \{(x \otimes y, H(x) + H(y) - H(f(D))) \mid x \in X^1, y \in X^2\}$$

which lies in a vector space of dimension $(|A^1| - 1) + (|A^2| - 1) + 1$. From Carathéodory's theorem, it can be obtained by a convex combination of at most $|A^1| + |A^2|$ points in S . Summing up, for each distribution D and correlation system Z s.t. $D(Z) = D$, there exists Z' with $|\text{supp } Z'| \leq K$, $D(Z') = D$ and $\Delta H(Z') = \Delta H(Z)$. It is plain that the set of correlation systems Z' s.t. $|\text{supp } Z'| \leq K$ and $D(Z') = D$ is a nonempty finite dimensional compact set and that the mapping ΔH is continuous on it. The maximum of ΔH is thus attained on this set. ■

The set of optimal correlation systems possesses a kind of consistency property. Roughly, one cannot find in the support of an optimal system, a sub-system which is not optimal. In geometric terms, if we denote by \mathcal{Z} the set of all correlation systems and $\mathcal{F}(Z)$ the minimal geometric face of the convex \mathcal{Z} containing Z , then the following lemma shows that if Z is optimal then any correlation system that belongs to $\mathcal{F}(Z)$ is also optimal (for a precise definition of the geometric face in infinite dimension, see e.g. [Lar04]).

Lemma 5 *Let Z be an optimal correlation system. If $Z = \lambda Z' + (1 - \lambda) Z''$ for some $0 < \lambda \leq 1$ and correlation systems Z', Z'' , then Z' is optimal.*

In particular, if $Z = \sum_{k=1}^K p_k \delta_{x_k \otimes y_k}$ is optimal, then for any k_1 and k_2 in $\{1, \dots, K\}$ such that $p_{k_1} + p_{k_2} > 0$, $\frac{p_{k_1}}{p_{k_1} + p_{k_2}} \delta_{x_{k_1} \otimes y_{k_1}} + \frac{p_{k_2}}{p_{k_1} + p_{k_2}} \delta_{x_{k_2} \otimes y_{k_2}}$ is optimal.

Proof. Assume that $Z = \lambda Z' + (1 - \lambda) Z''$ with $0 < \lambda \leq 1$ and that Z' is not optimal. There exists Z^* s.t. $D(Z^*) = D(Z')$ and $\Delta H(Z^*) > \Delta H(Z')$. Define $Z^0 = \lambda Z^* + (1 - \lambda) Z''$, then $D(Z^0) = D(Z)$ and $\Delta H(Z^0) - \Delta H(Z) = \lambda(\Delta H(Z^*) - \Delta H(Z'))$ contradicting the optimality of Z . ■

If we select for each D an optimal correlation system Z^D , the mapping U can be written as:

$$U(c) = \sup_{D: \Delta H(Z^D) \geq c} \pi(D),$$

so that to compute U , it suffices to compute one optimal Z^D for each distribution D . Note that Z^D is optimal for D if and only if:

$$\begin{aligned} \Delta H(Z^D) &= \max_{Z: D(Z)=D} \Delta H(Z) \\ &= \max_{Z: D(Z)=D} \left\{ \sum_k p_k (H(x_k) + H(y_k)) \right\} - H(f(D)), \end{aligned}$$

so that Z^D is optimal for D if and only if it is a solution of

$$\max_{Z: D(Z)=D} \sum_k p_k (H(x_k) + H(y_k)) \quad (P_D)$$

Note that this problem and therefore the notion of optimal correlation system, do not depend on payoffs and signals. A characterization of the solutions of (P_D) is thus a tool for computing $U(c)$ for any payoff function g and signaling function f . We establish the following properties on the value of (P_D) , which are independent of the size of the game.

Proposition 6 1. The mapping $\varphi: D \mapsto \text{value of } P_D$ is the smallest concave function on X^{12} that pointwise dominates the entropy function on X , i.e. $\varphi(x \otimes y) \geq H(x) + H(y)$ for each $x \otimes y \in X$.

2. For each D , $\varphi(D) \leq H(D)$ and $\varphi(D) = H(D)$ iff D is a product distribution.

3. φ is continuous on X^{12} .

Proof. (1) Let f be the bounded mapping $f : X^{12} \rightarrow \mathbb{R}$, such that

$$f(D) = \begin{cases} H(D) & \text{if } D \in X \\ 0 & \text{if } D \notin X \end{cases}$$

Then $\varphi = \text{cav } f$ the smallest concave function on X^{12} that is pointwise greater than f .

(2) If $D = \sum_k p_k x_k \otimes y_k$, by concavity of the entropy function, $H(D) \geq \sum_k p_k (H(x_k) + H(y_k))$, thus $H(D) \geq \varphi(D)$. Assume $D \in X$ i.e. $D = x \otimes y$, by point (1) $\varphi(x \otimes y) \geq H(x \otimes y)$ so that $\varphi(x \otimes y) = H(x \otimes y)$. If $D \notin X$, from proposition 4 there exists $(p_k, x_k, y_k)_k$ s.t. $D = \sum_k p_k x_k \otimes y_k$ and $\varphi(D) = \sum_k p_k (H(x_k \otimes y_k))$ and by strict concavity of the entropy function, $\varphi(D) < H(D)$.

(3) Since f is uppersemicontinuous and X^{12} is a polytope, we deduce from Laraki [Lar04] (theorem 1.16, proposition 2.1 and proposition 5.2) that φ is uppersemicontinuous. Also, since X^{12} is a polytope and φ is bounded and concave, we deduce from Rockafellar [Roc70] (theorem 10.2 and theorem 20.5) that φ is lowersemicontinuous. ■

3.2 Characterization when team members have two actions

We characterize optimal correlation systems when each team player possesses two actions. Hence assume from now on that $A^1 = A^2 = \{G, H\}$. We shall identify a mixed strategy x (resp. y) of player 1 (resp. 2) with the probability of playing G , i.e. to a number in the interval $[0, 1]$. We denote distributions $D \in X^{12}$ by:

$$D = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix},$$

where d_1 denotes the probability of the team's action profile (H, H) , d_2 the probability of (H, G) etc.

The following theorem shows that the optimal correlation system associated to any D is unique, contains at most two elements in its support, can be easily computed for a given distribution, and that the set of optimal correlation systems admits a simple parametrization.

Theorem 7 *For every $D \in X^{12}$, there exists a unique Z^D which is optimal for D . Moreover,*

- If $\det(D) = 0$, $Z^D = \delta_{x \otimes y}$ where

$$x = d_1 + d_2 \quad , \quad y = d_1 + d_3$$

- If $\det(D) < 0$, $Z^D = p\delta_{x \otimes y} + (1 - p)\delta_{y \otimes x}$ where

x and y are the two solutions of the second degree polynomial equation

$$X^2 - (2d_1 + d_2 + d_3)X + d_1 = 0$$

and

$$p = \frac{y - (d_1 + d_2)}{y - x}.$$

- If $\det(D) > 0$, $Z^D = p\delta_{1-x \otimes y} + (1 - p)\delta_{1-y \otimes x}$ where

x and y are the two solutions of the second degree polynomial equation

$$X^2 - (2d_3 + d_4 + d_1)X + d_3 = 0$$

and

$$p = \frac{y - (d_3 + d_4)}{y - x}$$

The proof is provided in section 5.1. Remark that each correlation system involves two points only in its support and that the parametrization of optimal correlation systems involves 3 parameters, matching the dimension of X^{12} . Note that proposition 4 only proves the existence of optimal correlation systems with $|A^1| + |A^2| = 4$ points in their support, thus described by 11 parameters.

4 Application to Γ_0

The optimal correlation systems are independent of the payoff function and signalling structures of the underlying game. We compute $U(c)$ for Γ_0 introduced in section 2.1. We introduce a family of correlation systems of particular interest.

Notation 8 For $x \in [0, 1]$ let $Z_x = \frac{1}{2}\delta_{x \otimes x} + \frac{1}{2}\delta_{(1-x) \otimes (1-x)}$.

It follows from theorem 7 that each Z_x is optimal. Actually, $(Z_x)_x$ is the family of correlation systems associated to probability measures that put equal weights on (G, G) and on (H, H) , and equal weights on (G, H) and on (H, G) . Against each Z_x , player 3 is thus indifferent between his two actions and therefore,

$$\pi(D(Z_x)) = \frac{1}{2}(x^2 + (1-x)^2).$$

For each $k = 1, 2$, $H(x_k) = H(y_k) = h(x)$ and the law of signals under $Z(x)$ is $(\frac{1}{2}, \frac{1}{2})$ thus,

$$\Delta H(Z_x) = 2h(x) - 1.$$

The following result, proved in section 5.3, shows that the map U can be obtained from the family $(Z_x)_x$.

Proposition 9 Consider the game Γ_0 . For any $c \in [-1, 1]$,

$$U(c) = \pi(D(Z_{x_c})) = \frac{1}{2}(x_c^2 + (1-x_c)^2)$$

with x_c the unique point in $[0, \frac{1}{2}]$ such that $2h(x_c) - 1 = c$. Moreover, U is concave.

It follows that the max min for the game Γ_0 is $U(0)$.

Corollary 10 The max min of the infinitely repeated game Γ_0 is:

$$v = \frac{1}{2}(x_0^2 + (1-x_0)^2)$$

where x_0 is the unique solution in $[0, \frac{1}{2}]$ of

$$-x \log(x) - (1-x) \log(1-x) = \frac{1}{2}$$

Numerically, $0.4020 < v < 0.4021$.

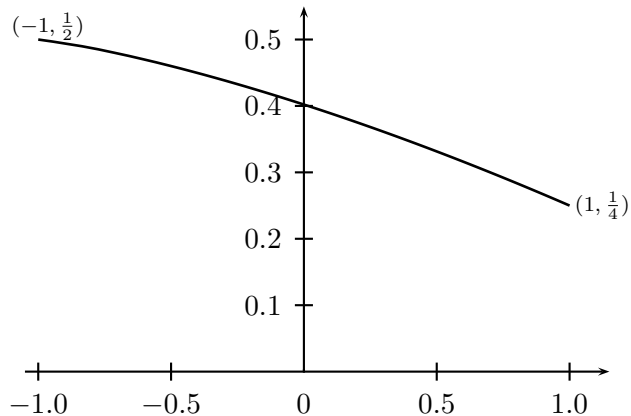


Figure 1: The graph of U

Remark 11 In contrast with a finite zero-sum stochastic game, the max min here is transcendental. A similar property holds for the asymptotic value of a repeated game with incomplete information on both sides (see Mertens and Zamir [MZ81]) and of a “Big Match” with incomplete information on one side (see Sorin [Sor84]).

5 Proofs of the main results

5.1 Proof of theorem 7

For each integer m , let $C_m(D)$ be the set of set vectors $(p_k, x_k, y_k)_{k=1}^m$ where:

$$\left\{ \begin{array}{l} \forall k, p_k \geq 0, \sum_{k=1}^m p_k = 1, x_k \in X^1, y_k \in X^2 \\ \sum_{k=1}^m p_k x_k \otimes y_k = D \end{array} \right.$$

This set is clearly compact and the mapping

$$(p_k, x_k, y_k)_{k=1}^m \mapsto \sum_{k=1}^m p_k (H(x_k) + H(y_k))$$

is continuous on it. The problem (P_D) can thus be written as:

$$\sup_m \max_{C_m(D)} \sum_{k=1}^m p_k (H(x_k) + H(y_k)) \quad (P_D)$$

Denote by $(P_{m,D})$, $m \geq 2$, the second maximization problem where m is fixed:

$$\max_{C_m(D)} \sum_{k=1}^m p_k(h(x_k) + h(y_k)) \quad (P_{m,D})$$

5.1.1 Solving $(P_{2,D})$.

Given $D \in X^{12}$, a point in $C_2(D)$ is a vector $(p, (x_1, y_1), (x_2, y_2)) \in [0, 1]^5$ such that:

$$D = p \begin{bmatrix} x_1 y_1 & x_1(1 - y_1) \\ (1 - x_1)y_1 & (1 - x_1)(1 - y_1) \end{bmatrix} + (1-p) \begin{bmatrix} x_2 y_2 & x_2(1 - y_2) \\ (1 - x_2)y_2 & (1 - x_2)(1 - y_2) \end{bmatrix}$$

The problem $(P_{2,D})$ writes:

$$\max_{C_2(D)} p(h(x_1) + h(y_1)) + (1 - p)(h(x_2) + h(y_2)) \quad (P_{2,D})$$

We are concerned with the computation of the set of solutions:

$$\Lambda(D) := \operatorname{argmax}_{C_2(D)} p(h(x_1) + h(y_1)) + (1 - p)(h(x_2) + h(y_2))$$

The problem $(P_{2,D})$ is the maximization of a continuous function on a compact set, thus $\Lambda(D) \neq \emptyset$ if $C_2(D) \neq \emptyset$. We will use the following parametrization: for $D = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$, set $r = d_1 + d_2$, $s = d_1 + d_3$ and $t = d_1$. The vector $(p, (x_1, y_1), (x_2, y_2)) \in [0, 1]^5$ is in $C_2(D)$ if and only if:

$$\begin{cases} px_1 + (1 - p)x_2 & = & r \\ py_1 + (1 - p)y_2 & = & s \\ px_1 y_1 + (1 - p)x_2 y_2 & = & t \end{cases}$$

Note that $\det(D) := d_1 d_4 - d_2 d_3 = t - rs$.

The remainder of this section is devoted to the proof of the following characterization of $\Lambda(D)$:

Proposition 12 (A) If $\det(D) = 0$, then

$$\begin{aligned}\Lambda(D) &= \{(p, (r, s), (r, s)) : p \in [0, 1]\} \\ &\cup \{(1, (r, s), (y_1, y_2)) : (y_1, y_2) \in [0, 1]^2\} \\ &\cup \{(0, (x_1, x_2), (r, s)) : (x_1, x_2) \in [0, 1]^2\}\end{aligned}$$

(B) If $\det(D) < 0$,

$$\Lambda(D) = \left\{ \left(\frac{\beta - r}{\beta - \alpha}, (\alpha, \beta), (\beta, \alpha) \right); \left(\frac{\alpha - r}{\alpha - \beta}, (\beta, \alpha), (\alpha, \beta) \right) \right\}$$

where α and β are the two solutions of:

$$X^2 - (2d_1 + d_2 + d_3)X + d_1 = 0.$$

(C) If $\det(D) > 0$,

$$\Lambda(D) = \left\{ \left(\frac{\beta - (1 - r)}{\beta - \alpha}, (1 - \alpha, \beta), (1 - \beta, \alpha) \right); \left(\frac{\alpha - (1 - r)}{\alpha - \beta}, (1 - \beta, \alpha), (1 - \alpha, \beta) \right) \right\}$$

where α and β are the two solutions of:

$$X^2 - (2d_3 + d_4 + d_1)X + d_3 = 0.$$

Proof of proposition 12. Remark that in (A), all solutions correspond to the same correlation system. The same applies to (B) and (C), in which both elements of $\Lambda(D)$ induce the same correlation system. Solutions of $(P_{2,D})$ always lead to a unique correlation system.

Point (A). The formula given in proposition 12 for $\Lambda(D)$ clearly defines a subset of $C_2(D)$. Note that $\det(D) = 0$ if and only if $D = r \otimes s$. (A) follows then directly from point (3) of lemma 6.

Points (B) and (C). We first show that these cases are deduced from one another by symmetry. Take a distribution $D = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$ and a point $(p, (x_1, y_1), (x_2, y_2))$ in $\Lambda(D)$. Let then $D' = \begin{bmatrix} d_3 & d_4 \\ d_1 & d_2 \end{bmatrix}$ and remark that:

- $\det(D') = -\det(D)$
- $(p, (1 - x_1, y_1), (1 - x_2, y_2)) \in \Lambda(D')$.

Remark also that the two solutions given in proposition 12 for case (C) are the symmetric of the solutions for case (B). We thus need to prove (B) only. Since α and β are solutions of:

$$X^2 - (2d_1 + d_2 + d_3)X + d_1 = 0.$$

we have $\alpha + \beta = r + s$ and $\alpha\beta = t$. Thus α , β , $\frac{\beta-r}{\beta-\alpha}$ and $\frac{r-\beta}{\alpha-\beta}$ are in $[0, 1]$. One then easily verifies that:

$$\begin{cases} \frac{\beta-r}{\beta-\alpha}\alpha + \frac{r-\beta}{\alpha-\beta}\beta & = r \\ \frac{\beta-r}{\beta-\alpha}\beta + \frac{r-\beta}{\alpha-\beta}\alpha & = s \\ \frac{\beta-r}{\beta-\alpha}\alpha\beta + \frac{r-\beta}{\alpha-\beta}\beta\alpha & = r \end{cases}$$

The solutions given in proposition 12 for case (B) are thus in $C_2(D)$ which is therefore non empty. In particular, any 2×2 joint distribution can be decomposed as a convex combination of two independent distributions.

We solve now the case where D is in the boundary of X^{12} .

Case 1. D is in the boundary.

Assuming $\det(D) < 0$, we get either:

$$D = D_1 = \begin{bmatrix} 0 & r \\ s & 1 - r - s \end{bmatrix}$$

or

$$D = D_2 = \begin{bmatrix} 1 - r - s & s \\ r & 0 \end{bmatrix}$$

with $rs > 0$. We solve for D_1 , the other case being similar. The vector $(p, (x_1, y_1), (x_2, y_2))$ is in $\Lambda(D_1)$ if and only if

$$\begin{aligned} px_1 + (1 - p)x_2 &= r \\ py_1 + (1 - p)y_2 &= s \\ px_1y_1 + (1 - p)x_2y_2 &= 0 \end{aligned}$$

Since D is not the product of its marginals, necessarily $p \in]0, 1[$, and $x_1 y_1 = x_2 y_2 = 0$. We assume wlog. $x_1 = 0$. We get then $x_2 = \frac{r}{1-p} \neq 0$, $y_2 = 0$, and $y_1 = \frac{s}{p}$. The problem (P_{2,D_1}) is then reduced to maximizing over $p \in (0, 1)$ the expression:

$$ph\left(\frac{s}{p}\right) + (1-p)h\left(\frac{r}{1-p}\right)$$

A solution in $(0, 1)$ exists, from the non emptiness of $\Lambda(D_1)$. The first order condition writes:

$$h\left(\frac{s}{p}\right) - \frac{s}{p}h'\left(\frac{s}{p}\right) = h\left(\frac{r}{1-p}\right) - \frac{r}{1-p}h'\left(\frac{r}{1-p}\right)$$

The map $f: (0, 1) \rightarrow \mathbb{R}$ given by $f(x) = h(x) - xh'(x)$ has derivative $f'(x) = -xh''(x) > 0$, hence is strictly increasing. Thus, the first order condition is equivalent to $\frac{r}{1-p} = \frac{s}{p}$, or $p = \frac{s}{r+s}$. We have thus shown:

$$\Lambda(D_1) = \left\{ \left(\frac{s}{r+s}, 0, r+s, r+s, 0 \right); \left(\frac{r}{r+s}, r+s, 0, 0, r+s \right) \right\}$$

Case 2. D is interior.

We assume now that $\min_{i \in \{1, \dots, 4\}}(d_i) > 0$. The proof is organized in a serie of lemmata. Lemma 13 proves that all solutions are interior. Therefore they must verify a first order condition. First order equations are established in lemma 14. Lemma 15 studies the solutions of the first order equations and lemma 16 shows unicity of those solutions. We conclude the proof with lemma 17.

We prove now that any solution of $(P_{2,D})$ is interior. This is due to the fact that the entropy function has infinite derivative at the boundary.

Lemma 13 *If $\min_{i \in \{1, \dots, 4\}}(d_i) > 0$ and $\det(D) \neq 0$ then $\Lambda(D) \subset (0, 1)^5$.*

Proof. We prove that elements of $\Lambda(D)$ are interior. Take a point $Z = (p, (x_1, y_1), (x_2, y_2))$ in $C_2(D)$. Since $\det(D) \neq 0$, $0 < p < 1$. We show that if $x_1 = 0$, Z is not optimal for $(P_{2,D})$. The proof is completed by symmetry. We assume thus $x_1 = 0$ and construct a correlation system

$Z^\varepsilon = (p^\varepsilon, (x_1^\varepsilon, y_1^\varepsilon), (x_2^\varepsilon, y_2^\varepsilon))$ in $C_2(D)$ as follows. Since $Z \in C_2(D)$:

$$\begin{cases} (1-p)x_2 & = r \\ py_1 + (1-p)y_2 & = s \\ (1-p)x_2y_2 & = t \end{cases}$$

Take $\varepsilon > 0$ and let:

$$\begin{cases} p^\varepsilon & = p + \varepsilon \\ x_1^\varepsilon & = \left(1 - \frac{p}{p^\varepsilon}\right)x_2 \\ x_2^\varepsilon & = x_2 \\ y_1^\varepsilon & = y_1 \\ y_2^\varepsilon & = \frac{1-p}{1-p^\varepsilon}y_2 - \frac{p^\varepsilon-p}{1-p^\varepsilon}y_1 \end{cases}$$

Since $t = (1-p)x_2y_2 \neq 0$, there exists $\varepsilon_0 > 0$ such that $Z^\varepsilon \in [0, 1]^5$ for $0 < \varepsilon \leq \varepsilon_0$. A simple computation shows that Z^ε is in $C_2(D)$. We now compare the objective function of $(P_{2,D})$ at Z^ε and at Z .

$$\begin{aligned} & (p^\varepsilon (h(x_1^\varepsilon) + h(y_1^\varepsilon)) + (1-p^\varepsilon) (h(x_2^\varepsilon) + h(y_2^\varepsilon))) \\ & - (p [h(x_1) + h(y_1)] + (1-p) [h(x_2) + h(y_2)]) \\ = & qh(x_1^\varepsilon) + (1-p^\varepsilon)h(y_2^\varepsilon) - (1-p)h(y_2) \\ = & (p+\varepsilon)h\left(\left(1 - \frac{p}{p+\varepsilon}\right)x_2\right) + (1-p-\varepsilon)h\left(\frac{1-p}{1-p-\varepsilon}y_2 - \frac{\varepsilon}{1-p-\varepsilon}y_1\right) - (1-p)h(y_2) \\ = & ph(\varepsilon x_2) + (1-p)h\left(y_2 - \frac{\varepsilon}{1-p}y_1\right) - (1-p)h(y_2) + o(\varepsilon) \\ = & ph(\varepsilon x_2) - \varepsilon y_1 h'(y_2) + o(\varepsilon) \\ = & p[-\varepsilon x_2 \ln(\varepsilon x_2) - (1-\varepsilon x_2) \ln(1-\varepsilon x_2)] - \varepsilon y_1 h'(y_2) + o(\varepsilon) \\ = & \varepsilon [-px_2 \ln(\varepsilon x_2) - y_1 h'(y_2) + x_2 + o(1)] \\ > & 0 \end{aligned}$$

for ε small enough. ■

Solutions of $(P_{2,D})$ being interior, they must verify first order conditions. Given x and y in $(0, 1)$, recall that the Kullback distance $d_K(x \| y)$ of x with respect to y is defined by:

$$d_K(x \| y) = -x \log \frac{x}{y} - (1-x) \log \frac{1-x}{1-y}$$

A direct computation shows:

$$d_K(x \| y) = h(y) - h(x) - h'(y)(y - x),$$

where h' denotes the derivative of h .

Lemma 14 *Suppose that $\min_i(d_i) > 0$ and $\det(D) \neq 0$. If $(p, x_1, y_1, x_2, y_2) \in \Lambda(D)$ then :*

$$\begin{cases} d_K(x_2 \| x_1) = d_K(y_1 \| y_2) \\ d_K(x_1 \| x_2) = d_K(y_2 \| y_1) \end{cases} \quad (E)$$

Proof. The Lagrangian of $(P_{2,D})$ writes:

$$\begin{aligned} \mathcal{L}(p, x_1, y_1, x_2, y_2, \alpha, \beta, \gamma) &= p(h(x_1) + h(y_1)) + (1 - p)(h(x_2) + h(y_2)) \\ &\quad + \alpha(px_1 + (1 - p)x_2 - r) + \beta(py_1 + (1 - p)y_2 - s) \\ &\quad + \gamma(px_1y_1 + (1 - p)x_2y_2 - t) \end{aligned}$$

The partial derivatives are

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial p} = (h(x_1) + h(y_1)) - (h(x_2) + h(y_2)) + \alpha(x_1 - x_2) + \beta(y_1 - y_2) + \gamma(x_1y_1 - x_2y_2) \\ \frac{\partial \mathcal{L}}{\partial x_1} = p(h'(x_1) + \alpha + \gamma y_1) \\ \frac{\partial \mathcal{L}}{\partial x_2} = (1 - p)(h'(x_2) + \alpha + \gamma y_2) \\ \frac{\partial \mathcal{L}}{\partial y_1} = p(h'(y_1) + \beta + \gamma x_1) \\ \frac{\partial \mathcal{L}}{\partial y_2} = (1 - p)(h'(y_2) + \beta + \gamma x_2) \end{cases}$$

If $(p, x_1, y_1, x_2, y_2) \in \Lambda(D)$, there exists (α, β, γ) such that:

$$\begin{cases} (h(x_1) + h(y_1)) - (h(x_2) + h(y_2)) + \alpha(x_1 - x_2) + \beta(y_1 - y_2) + \gamma(x_1y_1 - x_2y_2) = 0 & (E1) \\ h'(x_1) + \alpha + \gamma y_1 = 0 & (E2) \\ h'(x_2) + \alpha + \gamma y_2 = 0 & (E3) \\ h'(y_1) + \beta + \gamma x_1 = 0 & (E4) \\ h'(y_2) + \beta + \gamma x_2 = 0 & (E5) \end{cases}$$

The combination of equations $(E1) - x_1 \times (E2) + x_2 \times (E3)$ gives:

$$(h(x_1) + h(y_1)) - (h(x_2) + h(y_2)) = x_1 h'(x_1) - x_2 h'(x_2) - \beta(y_1 - y_2) \quad (1)$$

The combination $y_1((E4) - (E5)) + (x_1 - x_2)(E_2)$ writes:

$$y_1 (h'(y_1) - h'(y_2)) = h'(x_1)(x_1 - x_2) + \alpha(x_1 - x_2) \quad (2)$$

Equations (1) and (2) give:

$$h(x_1) - h(x_2) - h'(x_1)(x_1 - x_2) = h(y_2) - h(y_1) - h'(y_2)(y_2 - y_1)$$

which rewrites:

$$d_K(x_2 \| x_1) = d_K(y_1 \| y_2)$$

Similarly we obtain:

$$d_K(x_1 \| x_2) = d_K(y_2 \| y_1)$$

■

We give now the solutions of the equations (E).

Lemma 15 *Assume $d_K(x \| a) = d_K(b \| y)$ and $d_K(a \| x) = d_K(y \| b)$. Then one of the following holds:*

$$(F1) \quad x = b, y = a;$$

$$(F2) \quad x = 1 - b, y = 1 - a;$$

$$(F3) \quad x = a, y = b.$$

Proof. Fix a and b in $(0, 1)$. We need to solve the system:

$$\begin{cases} d_K(x \| a) - d_K(b \| y) = 0 \\ d_K(a \| x) - d_K(y \| b) = 0 \end{cases} \quad (S)$$

It is immediate to check that (F1), (F2), and (F3) are solutions of (S).

Letting $S(x, y) = (d_K(x \| a) - d_K(b \| y), d_K(a \| x) - d_K(y \| b))$, the Jacobian $J(x, y)$ of S writes:

$$\begin{aligned} J(x, y) &= \det \begin{pmatrix} \ln\left(\frac{x}{1-x}\right) - \ln\left(\frac{a}{1-a}\right) & \frac{1-a}{1-x} - \frac{a}{x} \\ \frac{1-b}{1-y} - \frac{b}{y} & \ln\left(\frac{y}{1-y}\right) - \ln\left(\frac{b}{1-b}\right) \end{pmatrix} \\ &= \ln \frac{x(1-a)}{a(1-x)} \times \ln \frac{y(1-b)}{b(1-y)} - \frac{(x-a) \times (y-b)}{x(1-x)y(1-y)} \end{aligned}$$

since for all $z > 1$, $0 < \ln(z) < z - 1$, if $x > a$ and $y > b$ then

$$0 < \ln \frac{x(1-a)}{a(1-x)} < \frac{x(1-a)}{a(1-x)} - 1 = \frac{x-a}{1-x} < \frac{x-a}{x(1-x)}$$

and

$$0 < \ln \frac{y(1-b)}{b(1-y)} < \frac{y(1-b)}{b(1-y)} - 1 = \frac{y-b}{1-y} < \frac{y-b}{y(1-y)}$$

Hence, on the domain $\{x > a, y > b\}$ one has:

$$\ln \frac{x(1-a)}{a(1-x)} \times \ln \frac{b(1-y)}{y(1-b)} < \frac{x-a}{1-x} \times \frac{y-b}{1-y} < \frac{(x-a) \times (b-y)}{x(1-x)y(1-y)}.$$

thus $J(x, y) < 0$ on the domain $\{x > a, y > b\}$. The mappings $x \mapsto d_K(x \| a) := f_a(x)$ and $y \mapsto d_K(b \| y) := g_b(y)$ are differentiable and strictly increasing on the intervals $(a, 1)$ and $(b, 1)$ respectively and setting $F(x) := g_b^{-1} \circ f_a(x) - f_b^{-1} \circ g_a(x)$, $S(x, y) = 0$ if and only if $F(x) = 0$ and $y = g_b^{-1} \circ f_a(x)$. Then if $x_0 \in (a, 1)$ is such that $F(x_0) = 0$, we let $y_0 := g_b^{-1} \circ f_a(x_0) = f_b^{-1} \circ g_a(x_0) \in (b, 1)$ and $F'(x_0) = \frac{J(x_0, y_0)}{f'_b(y_0) \times g'_b(y_0)} < 0$, *i.e.* at a zero of F , $F'(x_0) < 0$. F admits thus at most one zero.

If $a + b < 1$, $(1 - b, 1 - a)$ is indeed a solution of (S) and we deduce:

D₁. If $a + b < 1$, then $(1 - b, 1 - a)$ is the unique solution of (S) on $\{x > a, y > b\}$.

Using $z - 1 < \ln(z) < 0$ for all $z < 1$, we deduce that $J(x, y) < 0$ on the domain $\{x < a, y < b\}$. We then obtain:

D₂. If $a + b > 1$, then $(1 - b, 1 - a)$ is the unique solution of (S) on $\{x < a, y < b\}$.

Similar arguments show that:

D₃. If $a < b$, then (b, a) is the unique solution to (S) on $\{x > a, y < b\}$.

D₄. If $a > b$, then (b, a) is the unique solution to (S) on $\{x < a, y > b\}$.

We are now in position to complete the proof of the lemma. First, if $(x - a)(y - b) = 0$ then (S) implies $x = a$ and $y = b$.

If $(x - a)(y - b) > 0$, we obtain $(x, y) = (1 - b, 1 - a)$ as follows:

- If $a + b \leq 1$:

- If $x < a$ and $y < b$ then $x + y < a + b \leq 1$. Apply D_1 reversing the roles of (x, y) and (a, b) .

- If $x > a, y > b$ and $a + b \neq 1$. Apply D_1 .
- If $x > a, y > b$ and $a + b = 1$ then $x + y > 1$. Apply D_2 , reversing the roles.

• If $a + b > 1$:

- If $x > a$ and $y > b$, then $x + y > a + b > 1$. Apply D_2 , reversing the roles.
- If $x < a$ and $y < b$, apply D_2 .

If $(x - a)(y - b) < 0$ we obtain $(x, y) = (b, a)$ as follows:

• If $a \leq b$:

- If $x < a$ and $y > b$ then $x < y$. Reverse the roles and apply D_3 .
- If $x > a, y < b$ and $a < b$, apply D_3 .
- If $x > a, y < b$ and $a = b$ then $x > y$. Reverse the roles and apply D_4 .

• If $a > b$:

- If $x > a$ and $y < b$ then $x > y$. Reversing the roles and apply D_4 .
- If $x < a$ and $y > b$, apply D_4 .

■

Lemma 16 1. If $\det(D) < 0$, solutions of $(P_{2,D})$ are of type (F1).

2. If $\det(D) > 0$, solutions of $(P_{2,D})$ are of type (F2).

3. If $\det(D) = 0$, solutions of $(P_{2,D})$ are of type (F3).

Proof. Let $(p, a, b) \in [0, 1]^3$, it is straightforward to check that:

1. $\det [p(a \otimes b) + (1 - p)(b \otimes a)] \leq 0$

2. $\det [p(a \otimes b) + (1 - p)[1 - b] \otimes [1 - a]] \geq 0$

The result follows then directly from lemma 15. ■

We now conclude the proof of proposition 12

Lemma 17 *Let D such that $\det(D) < 0$. Then*

$$\Lambda(D) = \left\{ \left(\frac{\beta - r}{\beta - \alpha}, \alpha, \beta, \beta, \alpha \right); \left(\frac{r - \alpha}{\beta - \alpha}, \beta, \alpha, \alpha, \beta \right) \right\}$$

where α and β are the two solutions of the equation $X^2 - (r + s)X + t = 0$.

Proof. Assuming $\det(D) < 0$, it follows from lemma 16 that any element of $\Lambda(D)$ is a tuple $(p, (x, y), (y, x))$, with:

$$\begin{cases} px + (1 - p)y & = r \\ py + (1 - p)x & = s \\ pxy + (1 - p)yx & = t \end{cases}$$

We deduce then:

$$\begin{cases} x + y & = r + s \\ xy & = t \end{cases}$$

so that x and y must be solutions of the equation: $X^2 - (r + s)X + t = 0$

and p is given by $p = \frac{y-r}{y-x}$. Note that:

$$\Delta = (r + s)^2 - 4t \geq 4(rs - t) = -4\det(D) > 0$$

Hence, this equation admits two distinct solutions α and β . ■

The proof of proposition 12 is thus complete.

5.2 Solving $(P_{m,D})$

To conclude the proof of theorem 7, we prove that for every $D \in X^{12}$, the value of $P_{m,D}$ and of $P_{2,D}$ are the same. Recall from lemma 5 that if $(p_k, x_k, y_k)_{k \in K}$ is optimal for $P_{m,D}$, then for any pair (k_1, k_2) s.t. $p_{k_1} + p_{k_2} > 0$, the correlation system $((\frac{p_{k_1}}{p_{k_1} + p_{k_2}}, x_{k_1}, y_{k_1}); (\frac{p_{k_2}}{p_{k_1} + p_{k_2}}, x_{k_2}, y_{k_2}))$ is optimal for the distribution it induces. We deduce the solutions of $(P_{m,D})$ and of (P_D) from the form of solutions of $(P_{2,D})$

Lemma 18 *Let $(p_k, x_k, y_k)_{k=1}^m \in C_m(D)$ such that for all k , $p_k > 0$.*

If $(p_k, x_k, y_k)_{k=1}^m$ is optimal for (P_D) then one of the following holds:

- $\forall k$, if $(x_k, y_k) \neq (x_1, y_1)$ then $(x_k, y_k) = (y_1, x_1)$
- $\forall k$, if $(x_k, y_k) \neq (x_1, y_1)$ then $(x_k, y_k) = (1 - y_1, 1 - x_1)$

Proof. Suppose that $(x_2, y_2) \neq (x_1, y_1)$. Since $(p_k, x_k, y_k)_{k=1, \dots, m}$ is optimal for (P_D) , $\left(\left(\frac{p_1}{p_1+p_2}, x_1, y_1\right), \left(\frac{p_2}{p_1+p_2}, x_2, y_2\right)\right)$ is an optimal correlation system. Then one has either $(x_2, y_2) = (y_1, x_1)$ or $(x_2, y_2) = (1 - y_1, 1 - x_1)$. Suppose wlog. that $(x_2, y_2) = (y_1, x_1)$. Let us prove that if $(x_k, y_k) \neq (x_1, y_1)$ then we have also $(x_k, y_k) = (y_1, x_1)$. If it was not the case, we must have $(x_k, y_k) = (1 - y_1, 1 - x_1)$. Thus we deduce that $(x_k, y_k) = (1 - x_2, 1 - y_2)$. This is compatible with the form of optimal correlation system (with $m = 2$), only if we have either $(1 - x_2, 1 - y_2) = (1 - y_2, 1 - x_2)$ or $(1 - x_2, 1 - y_2) = (y_2, x_2)$. This means that we must assume either $x_2 = y_2$ or $x_2 = 1 - y_2$. If $x_2 = y_2$ then, since $(x_2, y_2) = (y_1, x_1)$, we should have $x_1 = y_1$. This implies that $(x_2, y_2) = (x_1, y_1)$, a contradiction with our assumption that $(x_2, y_2) \neq (x_1, y_1)$. Now, if $x_2 = 1 - y_2$ we deduce that $(x_k, y_k) = (y_2, x_2)$ from which we get $(x_k, y_k) = (x_1, y_1)$, also in contradiction with our assumption. Hence, if $(x_2, y_2) = (y_1, x_1)$ then $\forall k$, if $(x_k, y_k) \neq (x_1, y_1)$ one has $(x_k, y_k) = (y_1, x_1)$.

■

This ends the proof of theorem 7.

5.3 Proof of proposition 9

We use theorem 7 to solve the problem:

$$U(c) = \max_{D: \Delta H(Z^D) \geq c} \pi(D)$$

for the game Γ_0 .

Definition 19 *A correlation system Z is dominated for Γ_0 if there exists Z' such that $\pi(D(Z')) \geq \pi(D(Z))$ and $\Delta H(Z') \geq \Delta H(Z)$ with at least one strict inequality. Z is undominated otherwise.*

From theorem 7, undominated correlation systems must be of the form $p\delta_{x \otimes y} + (1 - p)\delta_{y \otimes x}$ or $p\delta_{x \otimes y} + (1 - p)\delta_{1-y \otimes 1-x}$. The next lemma shows that the first family of solutions is dominated.

Lemma 20 Given $Z = p\delta_{x\otimes y} + (1-p)\delta_{y\otimes x}$, let $Z' = \delta_{x\otimes y}$ and $Z'' = \delta_{y\otimes x}$. Then:

1. $\pi(D(Z)) = \pi(D(Z')) = \pi(D(Z''))$
2. $\Delta H(Z) \leq \max(\Delta H(Z'), \Delta H(Z''))$ with strict inequality if $x \neq y$ and $0 < p < 1$.

Proof. For point (1), the common value is $\min(xy, (1-x)(1-y))$. Point (2) follows from the formulas $\Delta H(Z) = h(x) + h(y) - h(px + (1-p)y)$, $\Delta H(Z') = h(x) + h(y) - h(x)$, $\Delta H(Z'') = h(y) + h(x) - h(y)$ and the strict concavity of h . ■

We search now solutions among the family of optimal correlation systems $p\delta_{x\otimes y} + (1-p)\delta_{1-y\otimes 1-x}$.

Lemma 21 Let $Z = p\delta_{x\otimes y} + (1-p)\delta_{1-y\otimes 1-x}$, $0 < p < 1$ and $x \neq 1-y$. If Z is undominated for Γ_0 , then $p = \frac{1}{2}$.

Proof. Denote the distribution induced by Z , $D(Z) = \begin{bmatrix} d_1(Z) & d_2(Z) \\ d_3(Z) & d_4(Z) \end{bmatrix}$

Assuming $x \neq 1-y$, $p = \frac{1}{2}$ is equivalent to $d_1(Z) = d_4(Z)$. Assume by contradiction that $d_1(Z) \neq d_4(Z)$ and by symmetry $d_1(Z) < d_4(Z)$. The Lagrangian of the maximization problem,

$$\begin{cases} \max & \pi(D(Z)) \\ Z = ((p, x, y); (1-p, 1-y, 1-x)) \\ \Delta H(Z) \geq c \end{cases}$$

writes :

$$\mathcal{L} = pxy + (1-p)(1-x)(1-y) - \alpha(h(x) + h(y) - h(px + (1-p)(1-y)))$$

Let $\tilde{y} = 1-y$ and $z = px + (1-p)\tilde{y}$:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial p} = & (x - \tilde{y})(1 - \alpha h'(z)) \\ \frac{\partial \mathcal{L}}{\partial x} = & -\tilde{y} + p + \alpha(h'(x) - ph'(z)) \\ \frac{\partial \mathcal{L}}{\partial y} = & x - 1 + p + \alpha(-h'(\tilde{y}) + (1-p)ph'(z)) \end{cases}$$

so that optimality of Z implies:

$$\begin{cases} h'(z) &= \frac{1}{\alpha} \\ \tilde{y} &= \frac{h'(x)}{h'(z)} \\ x &= \frac{h'(\tilde{y})}{h'(z)} \end{cases}$$

From the first two conditions we deduce that $h'(x)h'(\tilde{y}) \geq 0$, hence x and \tilde{y} lie on the same side of $\frac{1}{2}$. But then $|h'(z)| \geq |h'(x)|$ and $|h'(z)| \geq |h'(\tilde{y})|$ is inconsistent with z lying in the strict interval of extremities x and \tilde{y} : $0 < p < 1$, $x \neq \tilde{y}$. ■

Lemma 22 *Let $Z = \frac{1}{2}\delta_{x \otimes y} + \frac{1}{2}\delta_{1-y \otimes 1-x}$, with $x \neq 1-y$. If Z is not dominated for Γ_0 , then $x = y$.*

Proof. Let $z = \frac{x+y}{2}$, and $Z' = ((\frac{1}{2}, z, z), (\frac{1}{2}, 1-z, 1-z))$. We prove that Z' dominates Z in G if $x \neq y$. For payoffs, direct computation leads $\pi(D(Z')) - \pi(D(Z)) = (\frac{x+y}{2})^2$. For entropy variations, let ψ be defined by $\psi(x, y) = h(x) + h(y) - h(\frac{x+1-y}{2})$. Then $\Delta H(Z) = \psi(x, y) = \psi(y, x)$ and $\Delta H(Z') = \psi(\frac{x+y}{2}, \frac{x+y}{2})$. Inequality $\psi(\frac{x+y}{2}, \frac{x+y}{2}) > \frac{\psi(x, y) + \psi(y, x)}{2}$ will follow from the strict concavity of ψ . The Jacobian matrix of ψ is:

$$J = \begin{pmatrix} h''(x) - \frac{1}{4}h''(\frac{x+1-y}{2}) & -\frac{1}{4}h''(\frac{x+1-y}{2}) \\ -\frac{1}{4}h''(\frac{x+1-y}{2}) & h''(y) - \frac{1}{4}h''(\frac{x+1-y}{2}) \end{pmatrix}$$

Then, $\text{trace}J = h''(x) + h''(y) - \frac{1}{4}h''(\frac{x+1-y}{2}) = h''(x) + h''(1-y) - \frac{1}{4}h''(\frac{x+1-y}{2})$ is negative since $h'' : t \mapsto -\frac{1}{\ln 2}(\frac{1}{t} + \frac{1}{1-t})$ is both concave and negative on $(0, 1)$. Brute force computation of $\det J$ shows:

$$\begin{aligned} \det J &= \frac{1}{(\ln 2)^2} \frac{(1-x)(1-y) + xy}{xy(1-x)(1-y)(1-x+y)(1-y+x)} \\ &> 0 \end{aligned}$$

Hence the strict concavity of ψ , and the claim follows. ■

We prove now proposition 9. From the two previous lemmas, it follows that an undominated correlation system is of the form $Z(x) = \frac{1}{2}\delta_{x \otimes x} + \frac{1}{2}\delta_{1-x \otimes 1-x}$ with $x \in [0, 1]$. The graph of $c \mapsto U(c)$ is thus the set:

$$C = \{(\Delta H(Z), \pi(D(Z))), Z = \frac{1}{2}\delta_{x \otimes x} + \frac{1}{2}\delta_{1-x \otimes 1-x} \text{ and } x \in [0, 1]\}$$

By symmetry one needs only to consider to $x \in [0, \frac{1}{2}]$, and letting $(s(x), t(x)) = (2h(x) - 1, \frac{1}{2}x^2 + \frac{1}{2}(1-x)^2)$, C is the parametric curve $\{(s(x), t(x)), x \in [0, \frac{1}{2}]\}$. Since the slope $\alpha(x)$ of C at $(s(x), t(x))$ is

$$\alpha(x) = \frac{dt(x)/dx}{ds(x)/dx} = \frac{1-2x}{\log(1-x) - \log(x)}$$

and

$$\alpha'(x) = \frac{2x-1 + 2x(1-x)\ln(1-\frac{1}{x})}{\ln(2)x(1-x)(\log(1-x) - \log(x))^2}$$

The numerator of this expression has derivative $(1-2x)\ln(\frac{1}{x}-1) > 0$, and takes the value 0 at $x = \frac{1}{2}$, hence it is nonnegative and so is $\alpha'(x)$. We conclude that C is concave and that $U(c) = \pi(D(Z(x_c)))$ with $\Delta H(Z(x_c)) = 2h(x_c) - 1 = c$ and $\text{cav} U(0) = U(0)$. This value is $\frac{1}{2}x^2 + \frac{1}{2}(1-x)^2$, where $0 < x < 1$ solves $h(x) = \frac{1}{2}$. Numerical resolution yields $0.1100 < x < 0.1101$ and $0.4020 < \frac{1}{2}x^2 + \frac{1}{2}(1-x)^2 < 0.4021$.

6 Concluding remarks

6.1 On other games

The function $U(c)$ is determined by the one-shot game and the signalling function. Since we deal with the computation of $\text{cav} U(0)$ two cases may arise: either $\text{cav} U(0) = U(0)$ (for example, if U is concave) or $\text{cav} U(0) > U(0)$ (if there exists two correlation systems Z_1, Z_2 and $\lambda \in (0, 1)$ s.t. $\lambda\pi(D(Z_1)) + (1-\lambda)\pi(D(Z_2)) > U(0)$ and $\lambda\Delta H(Z_1) + (1-\lambda)\Delta H(Z_2) \geq 0$).

In the previous section we have shown that the map U corresponding to Γ_0 is concave. Goldberg [Gol03] provides an example of the second case. Consider the game where payoffs for players 1 and 2 are given by:

	G	H	
G	1	0	
H	3	1	
	G		

	G	H
G	1	3
H	0	1
	G	H

and the signalling function f is:

	<i>G</i>	<i>H</i>
<i>G</i>	<i>s</i>	<i>s''</i>
<i>H</i>	<i>s'</i>	<i>s</i>
	<i>G</i>	

The maxmin of the one-shot game is $\frac{5}{4}$ and is obtained by the distribution $\frac{1}{2} \otimes \frac{1}{2}$. Allowing for correlation, the maxmin is $\frac{3}{2}$ and is obtained by the distribution $\frac{1}{2}0 \otimes 1 + \frac{1}{2}1 \otimes 0$.

Relying on theorem 7, Goldberg shows that U is convex so that its concavification is linear, thus $\text{cav } U(0) = \frac{4}{3} = \frac{2}{3}\pi(D(Z')) + \frac{1}{3}\pi(D(Z''))$ where $Z' = \delta_{\frac{1}{2} \otimes \frac{1}{2}}$ and $Z'' = \frac{1}{2}\delta_{0 \otimes 1} + \frac{1}{2}\delta_{1 \otimes 0}$.

6.2 Extensions and open problems

The obvious way of extending our results is to consider larger sets of actions for the team players as well as larger teams. Assume that the team of players $\{1, \dots, n\}$ opposes player $n+1$ in a repeated game where each player $1 \leq i \leq n$ has perfect monitoring and player $n+1$ observes a signal $f(a^1, \dots, a^n)$. The characterization of the maxmin in [GT04b] is valid in this case. The definition of correlation systems and of optimal correlation systems extend naturally. A correlation system is a finite support distribution on the set of mixed strategies: $Z = \sum_k p_k \delta_{\otimes_i x_k^i}$ with $x_k^i \in \Delta(A^i)$. The correlation system is optimal if $D(Z) = D$ and Z solves:

$$\max_{Z: D(Z)=D} \sum_k p_k \left(\sum_i H(x_k^i) \right)$$

Then proposition 4, lemma 5 and proposition 6 perfectly extend.

Further, in the case of two team-players and general number of actions, one can show that lemma 14 extends as follows: if the correlation system $\alpha \delta_{x_1 \otimes y_1} + (1 - \alpha) \delta_{x_2 \otimes y_2}$ with $\alpha \in (0, 1)$ is optimal then

$$\begin{cases} d_K(x_2 \| x_1) = d_K(y_1 \| y_2) \\ d_K(x_1 \| x_2) = d_K(y_2 \| y_1) \end{cases},$$

where $d_K(x \| y)$ is the Kulback distance between x and y .

The generalization of theorem 7 to the general setup is an open problem.

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